WIR_21A_M152_H1

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NOTE ♯1: SUBSTITUTION AND AREA BETWEEN CURVES

Problem 1. Compute $\int \frac{\cos^{-1}(3x)}{\sqrt{1-9x^2}}dx.$

$u = 3x \Rightarrow \frac{du}{3} = \frac{3}{3}dx$

$\int \frac{1}{3}\frac{\cos^{-1}(u)}{\sqrt{1-u^2}}\,du$

$y = \cos^{-1}u \Rightarrow dy = \frac{-1}{\sqrt{1-u^2}}du$

$\int -\frac{1}{3}\,y\,dy = -\frac{1}{3}\frac{1}{2}y^2 + C = -\frac{1}{6}\left(\cos^{-1}(3x)\right)^2 + C$

Problem 2. Compute $\int e^{2x}\cos(e^{2x})dx.$

$u = e^{2x} \quad du = 2e^{2x}dx$

$\int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(e^{2x}) + C$

Problem 3. Compute $\int \frac{1}{\sin^2 x\sqrt{1-\cot x}} dx.$

$u = 1 - \cot x$

$du = + \csc^2 x \, dx =$

$du = \frac{+1}{\sin^2 x} dx$

$$\int \frac{1}{\sqrt{u}} du =$$

$$= \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{1-\cot x} + C$$

Problem 4. Compute $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}} dx.$

$u = \sqrt{x+1}$

$2\,du = \frac{\cancel{2}\,1}{\cancel{2}\sqrt{x+1}} dx$

$$\int 2e^u du = 2e^u + C =$$

$$= 2e^{\sqrt{x+1}} + C$$

Problem 5. Compute $\int (x^7)\sqrt{x^4+5}\,dx$.

$u = x^4 + 5$

$x^4 = u - 5$

$du = 4x^3\,dx$

$$\int x^4\sqrt{x^4+5}\;x^3dx$$

$$\int (u-5)\sqrt{u}\,\frac{1}{4}\,du =$$

$u^{1/2}(u^1 - 5) = u^{3/2} - 5u^{1/2}$

$$= \frac{1}{4}\int u^{3/2} - 5u^{1/2}\,du = \frac{1}{\cancel{4}_2}\left[\frac{\cancel{2}^1}{5}u^{5/2} - 5\,\frac{\cancel{2}^1}{3}u^{3/2}\right] + C =$$

$$= \frac{1}{10}(x^4+5)^{5/2} - \frac{5}{6}(x^4+5)^{3/2} + C$$

Problem 6. Compute $\int \frac{\sec\frac{1}{x^2}\tan\frac{1}{x^2}}{x^3}dx.$

$u = \frac{1}{x^2} = x^{-2}$

$du = -2x^{-3}\,dx =$

$du = \frac{-2}{x^3}\,dx$

$\frac{dx}{x^3} = \frac{du}{-2}$

$$\int \frac{1}{-2}\sec u\tan u\;du =$$

$$= -\frac{1}{2}\sec u + C = -\frac{1}{2}\sec\left(\frac{1}{x^2}\right) + C$$

Problem 7. Compute $\int \frac{x-x^3}{1+x^4}dx$.

$u = 1+x^4$

$\frac{du}{4} = \frac{4}{4}x^3\,dx$

$\int\left(\frac{x}{1+x^4} - \frac{x^3}{1+x^4}\right)dx$

$\int \frac{x}{1+(x^2)^2}dx - \int \frac{x^3}{1+x^4}dx$

$u = x^2$

$\frac{du}{2} = \frac{2}{2}x\,dx$

$\int \frac{1}{2}\frac{1}{1+u^2}du - \int \frac{1}{4}\frac{1}{u}du$

$\frac{1}{2}\arctan u - \frac{1}{4}\ln|u| + C$

$\frac{1}{2}\arctan(x^2) - \frac{1}{4}\ln|1+x^4| + C$

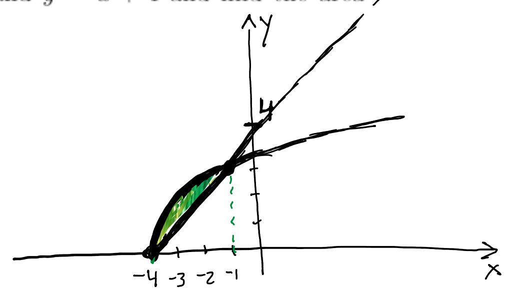
$\sqrt{3x+12} = x+4$

$3x+12 = x^2+8x+16$

$x^2+5x+4=0 \quad (x+4)(x+1)$

$3x+12=0 \Rightarrow x = -\frac{12}{3} = -4$

Problem 8. Sketch the region bounded by $y = \sqrt{3x+12}$ and $y = x+4$ and find the area between them.



$$\text{Area} = \int_{-4}^{-1} \sqrt{3x+12} - (x+4)\,dx =$$

$$= \int_{-4}^{-1} \sqrt{3x+12}\,dx - \int_{-4}^{-1} x+4\,dx$$

$u^{1/2}$

$u = 3x+12 \qquad du = 3dx$

$3(-1)+12=9$

$$\int_0^9 \frac{1}{3}\sqrt{u}\,du = \frac{1}{3}\,\frac{2}{3}\,u^{3/2}\Big|_0^9 =$$

$3\cdot(-4)+12=0$

$$= \frac{2}{9}\left(9^{3/2} - 0^{3/2}\right) = \frac{2}{\cancel{9}}\cdot \overset{3}{\cancel{27}} = 6$$

$\sqrt{9}^3 = 3^3$

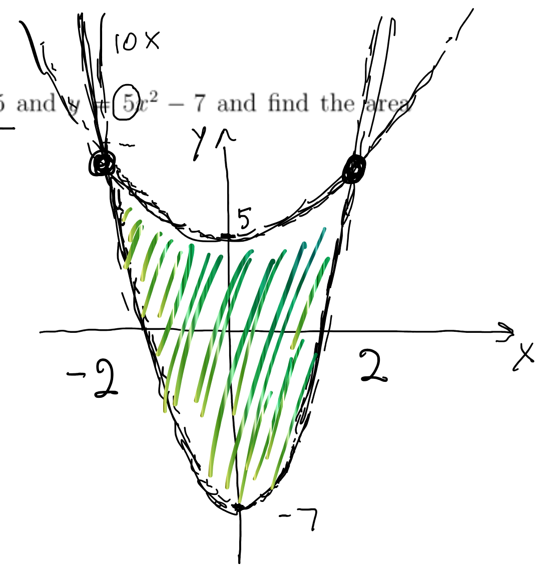
$$-\left(\frac{x^2}{2}+4x\right)\Big|_{-4}^{-1} = -\left[\frac{1}{2}-4-\left(\frac{16}{2}-16\right)\right] =$$

$$= -\left[12-\frac{15}{2}\right] = -\frac{24-15}{2} = -\frac{9}{2}$$

$$\text{Area} = 6 - \frac{9}{2} = \frac{12-9}{2} = \frac{3}{2}$$

4x

Problem 9. Sketch the region bounded by $y = 2x^2 + 5$ and $y = 5x^2 - 7$ and find the area between them.



$$2x^2 + 5 = 5x^2 - 7$$
$$12 = 3x^2$$
$$4 = x^2 \Rightarrow x = \pm 2$$

$$\text{Area} = \int_{-2}^{2} (2x^2+5) - (5x^2-7)\, dx =$$

$$= \int_{-2}^{2} (-3x^2 + 12)\, dx = -x^3 + 12x \Big|_{-2}^{2} =$$

$$= (-8 + 24) - (8 - 24) = 32.$$

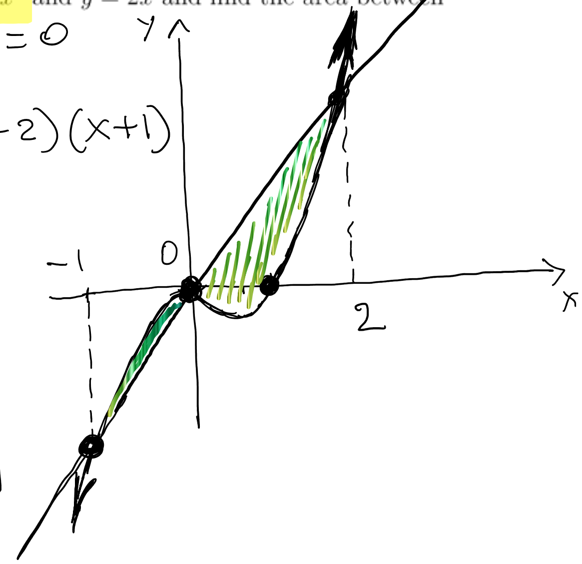
$y' = 3x^2 - 2x$
$= x(3x-2)$

$x^2(x-1)$

Problem 10. Sketch the region bounded by $y = x^3 - x^2$ and $y = 2x$ and find the area between them.

$x^3 - x^2 = 2x$

$x(x^2 - x - 2) = 0$ — $x = 0$; $(x-2)(x+1)$



$$\text{Area} = \int_{-1}^{2} |x^3 - x^2 - 2x|\, dx =$$

$$= \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx + \int_{0}^{2} (2x - x^3 + x^2)\, dx$$

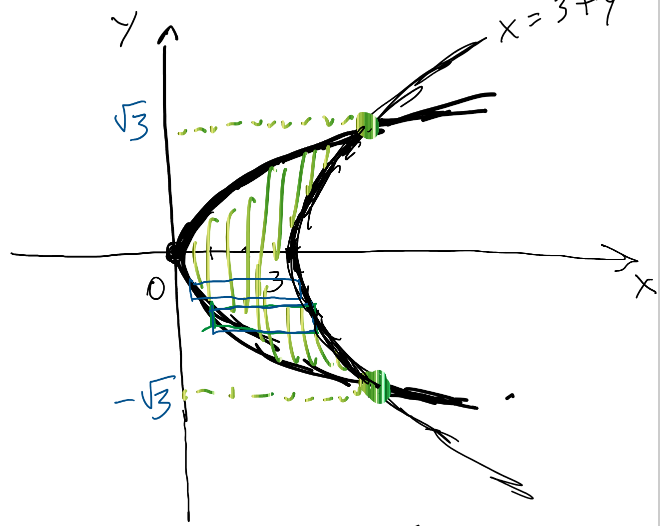
$$\left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right]_{0}^{2} =$$

$$= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) + \left(4 - \frac{\cancel{16}^{4}}{4} + \frac{8}{3} - 0\right) = \frac{-(3+4-12)}{12} + \left(\frac{8}{3}\right) =$$

$$= \frac{5}{12} + \frac{8}{3} = \frac{5+32}{12} = \frac{37}{12}$$

Problem 11. Sketch the region bounded by $x = 2y^2$ and $x = 3 + y^2$ and find the area between them.

$$2y^2 = 3 + y^2$$
$$y^2 = 3 \Rightarrow y = \pm\sqrt{3}$$

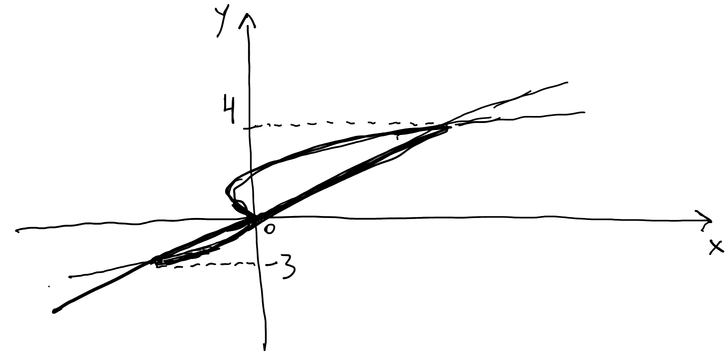


$$\text{Area} = \int x_R - x_L \, dy =$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \overbrace{(3+y^2) - (2y^2)}^{3-y^2} \, dy =$$

$$2\left[3y - \frac{1}{3}y^3\right]_0^{\sqrt{3}} = 2\left[3\sqrt{3} - \frac{1}{\cancel{3}}\cancel{3}\sqrt{3} - 0\right] = 4\sqrt{3}.$$

Problem 12. Sketch the region bounded by $x = y^3 - y^2$ and $x = 12y$ and find the area between them.



$y^3 - y^2 = 12y$

$y(y^2 - y - 12) = 0$

$y(y-4)(y+3)$ — $y=0$, $y=4$, $y=-3$

$$\text{Area} = \int_{-3}^{4} |x_R - x_L| \, dy$$

$$\int_{-3}^{0} (y^3 - y^2 - 12y)\, dy + \int_{0}^{4} (12y - y^3 + y^2)\, dy$$

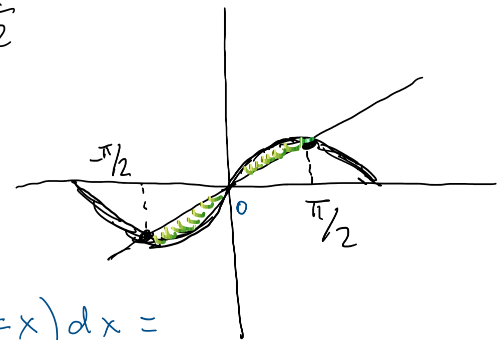
$$\left[\frac{1}{4}y^4 - \frac{1}{3}y^3 - 6y^2\right]_{-3}^{0} + \left[6y^2 - \frac{1}{4}y^4 + \frac{1}{3}y^3\right]_{0}^{4} =$$

$$= \left[0 - \left(\frac{81}{4} + \frac{27}{3} - 54\right)\right] + \left[6 * 16 - \frac{4^4}{4} + \frac{64}{3}\right] =$$

$$= \ldots = \frac{291}{4}$$

Problem 13. Sketch the region bounded by $y = \sin x$ and $x = \frac{2}{\pi}x$ and find the area between them.

$\sin x = \frac{2}{\pi} x$ $\qquad \sin\frac{\pi}{2} = 1 = \frac{2}{\pi}\frac{\pi}{2}$



Area $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sin x - \frac{2}{\pi} x \right| dx =$

$$= \int_{-\frac{\pi}{2}}^{0} \left(\frac{2}{\pi}x - \sin x\right) dx + \int_{0}^{\pi/2} \left(\sin x - \frac{2}{\pi}x\right) dx =$$

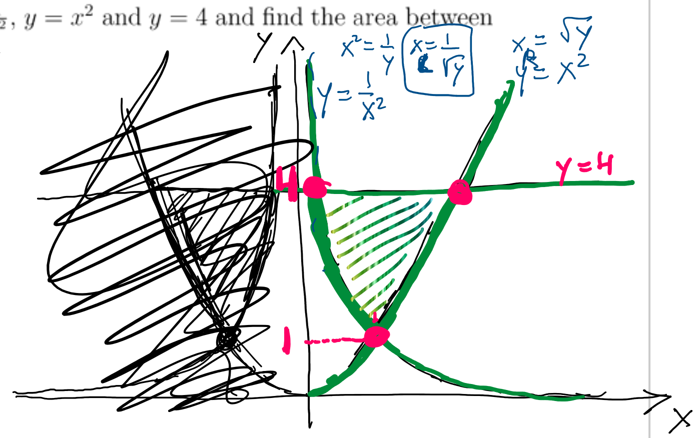
$$= \left[\frac{x^2}{\pi} + \cos x\right]_{-\frac{\pi}{2}}^{0} + \left[-\cos x - \frac{x^2}{\pi}\right]_{0}^{\pi/2} =$$

$$\left(0 + 1 - \frac{1}{\pi}\frac{\pi^2}{4} - 0\right) + \left(0 - \frac{1}{\pi}\frac{\pi^2}{4} - ((-1) - 0)\right)$$

$$1 - \frac{\pi}{4} + \left(-\frac{\pi}{4} + 1\right) = 2 - \frac{\pi}{2}$$

in the first quadrant

Problem 14. Sketch the region bounded by $y = \frac{1}{x^2}$, $y = x^2$ and $y = 4$ and find the area between them.



Area $= \int_1^4 (x_L - x_R)\, dy$

$$\int_1^4 \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right) dy = \int y^{1/2} - y^{-1/2}$$

$$\left[\frac{2}{3} y^{3/2} - 2\, y^{1/2}\right]_1^4 =$$

$$\frac{2}{3} \cdot 8 - 2 \cdot 2 - \left(\frac{2}{3} - 2\right) =$$

$$= \frac{16}{3} - 4 - \frac{2}{3} + 2 = \frac{14}{3} - 2 = \frac{14-6}{3} = \frac{8}{3}$$

$\frac{1}{x^2} = x^2 \Rightarrow 1 = x^4 \Rightarrow x = \pm 1$

$y = 1$

$$\sin(5x + 5x) = \underbrace{2 \sin 5x \cos 5x} = \cos 5x$$

$$\cancel{4} \sin 10x = \cancel{2} \cos 5x$$