LAST WIR

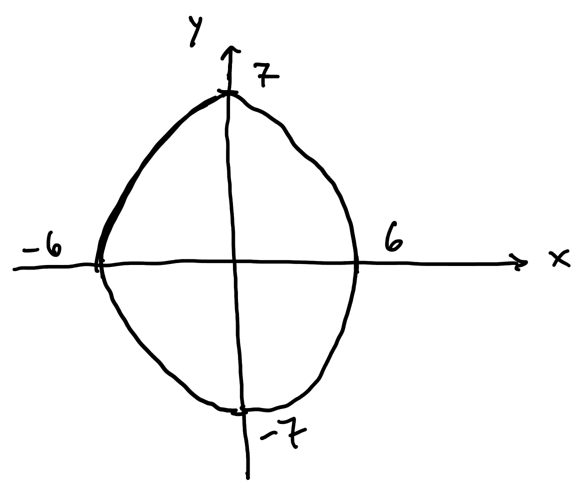
Problem 1. Sketch the curve traced out by $x = 6\cos t$, $y = 7\sin t$, $0 \le t \le 2\pi$.

Claim: The curve is an ellipse.

In fact, note that:

$$\frac{x^2}{36} + \frac{y^2}{49} = \frac{\cancel{36}\cos^2 t}{\cancel{36}} + \frac{\cancel{49}\sin^2 t}{\cancel{49}} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{49} = 1$$



$x' = 6 - 6t^2 \quad y' = 12t$

Problem 2. Find the length of $x = 6t - 2t^3$, $y = 6t^2$ for $0 \le t \le 1$.

$$\mathcal{L} = \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \text{, in this case}$$

$$= \int_0^1 \sqrt{(6-6t^2)^2 + (12t)^2}\, dt$$

$$36 - \overbrace{72t^2 + \underbrace{36t^4} + 144t^2}^{+72}$$

$$= \int_0^1 \sqrt{36 + 72t^2 + 36t^4}\, dt \qquad \sqrt{36(1 + 2t^2 + t^4)}$$

$$= \int_0^1 6\sqrt{1 + 2t^2 + t^4}\, dt$$

$$= \int_0^1 6\sqrt{(1+t^2)^2}\, dt$$

$$= \int_0^1 6(1+t^2)\, dt = 6\left[t + \frac{t^3}{3}\right]_0^1 = 6\left(1 + \frac{1}{3}\right) = \cancel{6}^{2}\left(\frac{4}{\cancel{3}_1}\right) = 8.$$

$x' =$ $t=-1$ $t=1$

Problem 3. Find the length of $x = 6t - 2t^3$, $y = 6t^2$ from $(-4, 6)$ to $(4, 6)$.

$$\left.\begin{array}{l} 6t - 2t^3 = -4 \\ 6t^2 = 6 \Rightarrow t = \pm 1 \end{array}\right\} t = -1$$

$$\mathcal{L} = \int_{-1}^{1} \sqrt{\quad \cdots \quad} = \cdots \text{ same as above}$$

$$6\left(t + \frac{t^3}{3}\right)\Bigg|_{-1}^{1} = 2 \cdot 8 = 16.$$

$a+$

Problem 4. Find the length of $x = e^{2t} + e^{-2t}$, $y = b - 4t$ for $0 \le t \le 1$.

$$x' = 2e^{2t} - 2e^{-2t} \qquad y' = -4$$

$$\mathcal{L} = \int_0^1 \sqrt{[2(e^{2t} - e^{-2t})]^2 + (-4)^2}\, dt$$

$$\sqrt{4(e^{4t} - 2\overbrace{e^{2t}e^{-2t}}^{\text{equals } 1} + e^{-4t}) + 16}$$

$$\sqrt{4e^{4t} - 8 + 4e^{-4t} + 16}$$

$$\sqrt{4e^{4t} + 8 + 4e^{-4t}}$$

$$\sqrt{4(e^{4t} + 2 + e^{-4t})}$$

$$\sqrt{4(e^{2t} + e^{-2t})^2}$$

$$= \int_0^1 2(e^{2t} + e^{-2t})\, dt$$

$$= \cancel{2}\left[\frac{1}{\cancel{2}} e^{2t} - \frac{1}{\cancel{2}} e^{-2t}\right]_0^1$$

$$= (e^2 - e^{-2}) - (e^0 - e^0)$$

$$= e^2 - \frac{1}{e^2}$$

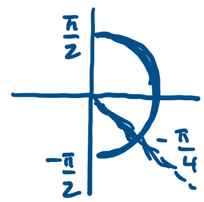
Problem 5. Find the length of $x = a + e^{2t} + e^{-2t}$, $y = b - 4t$ for $0 \le t \le 1$ for any real numbers a and b.

- The exact same as above because $x'(t)$ and $y'(t)$ and $0 \le t \le 1$ are the same!

Problem 6. Find the cartesian coordinates for $(1, \pi)$, $(2, 2\pi)$, $(3, \frac{\pi}{4})$, $(1, \frac{\pi}{6})$.

r	θ	$x = r\cos\theta$	$y = r\sin\theta$	(x, y)
1	π	$x = 1\ (-1)$	$y = 1 \cdot 0$	$(-1, 0)$
2	2π	$x = 2\ (1)$	$y = 2 \cdot 0$	$(2, 0)$
3	$\frac{\pi}{4}$	$x = 3\ \frac{\sqrt{2}}{2}$	$y = 3\ \frac{\sqrt{2}}{2}$	$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$
1	$\frac{\pi}{6}$	$x = 1\ \frac{\sqrt{3}}{2}$	$y = 1\ \frac{1}{2}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Problem 7. Find the polar coordinates for $(1, -1)$ and $(-1, \sqrt{32})$.



$r^2 = x^2 + y^2$ $\qquad \theta = \tan^{-1} \frac{y}{x}$

$r^2 = 1 + 1 = 2$

choose $r = +\sqrt{2}$ $\qquad \theta = \tan^{-1} \frac{-1}{1} = -\frac{\pi}{4}$

$\left(\sqrt{2}, -\frac{\pi}{4}\right)$ OR $\left(-\sqrt{2}, \frac{3\pi}{4}\right) \ldots$

$r^2 = 1 + 32 = 33$ $\qquad \theta = \tan^{-1} \frac{\sqrt{32}}{-1}$

$r = +\sqrt{33}$

$$\left(\sqrt{33}, \underbrace{\tan^{-1}\left(-\sqrt{32}\right)}_{\approx -79^\circ}\right)$$

Problem 8. Find a cartesian equation for $r = 2\cos\theta$.

$$(r)\quad r = 2\cos\theta \quad (r)$$

counterclockwise.

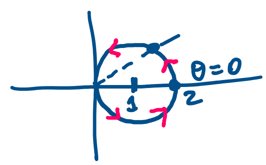
$$\underbrace{r^2} = 2\,\underbrace{r\cos\theta}$$

$$x^2 + y^2 = 2x$$

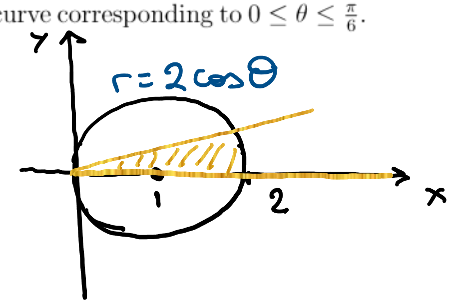
$$\underbrace{x^2 - 2x + 1} + y^2 = \underbrace{0 + 1}$$

$$(x-1)^2 + y^2 = 1$$

$$c\ (1, 0) \quad r = 1$$



Problem 9. Find the area of the region contained in the above curve corresponding to $0 \leq \theta \leq \frac{\pi}{6}$.



$$A = \int_0^{\pi/6} \frac{1}{2}[r(\theta)]^2 \, d\theta =$$

$$= \frac{1}{2} \int_0^{\pi/6} 4\cos^2\theta \, d\theta =$$

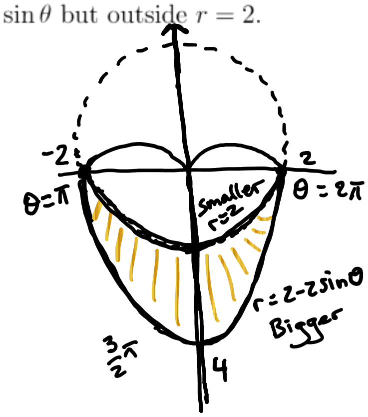
$$= \frac{1}{\cancel{2}} \int_0^{\pi/6} \cancel{4} \frac{1}{\cancel{2}} (1 + \cos 2\theta) \, d\theta$$

$$= \left[\theta + \frac{1}{2} \sin \underline{2\theta}\right]_0^{\pi/6} =$$

$$= \left(\frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2}\right) - (0 + 0)$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{4}.$$

Problem 10. Find the area inside $r = 2 - 2\sin\theta$ but outside $r = 2$.



$\int_{\pi}^{2\pi}$ OR $2\int_{\frac{3}{2}\pi}^{2\pi}$

$$A = \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2}\left[(2-2\sin\theta)^2 - (2)^2\right] d\theta$$

$$\int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2}\left(8 + 4\sin^2\theta - 8\sin\theta\right) d\theta$$

$4 - 8\sin\theta + 4\sin^2\theta + 4$

$$4 + 2\sin^2\theta - 4\sin\theta = 4 + 2\cdot\frac{1}{2}(1-\cos 2\theta) - 4\sin\theta$$

$$= 5 - \cos 2\theta - 4\sin\theta$$

integrate this

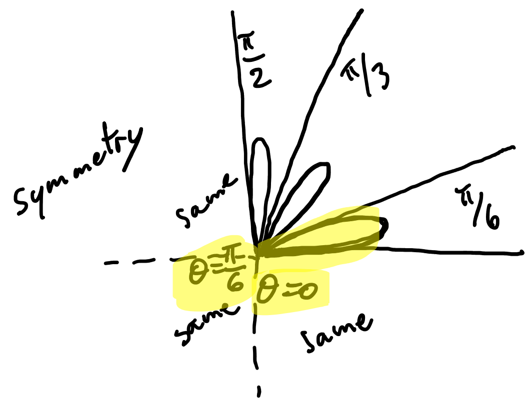
$\int$

$$\left[5\theta - \frac{1}{2}\sin 2\theta + 4\cos\theta\right]_{\frac{3\pi}{2}}^{2\pi} =$$

$$10\pi - 0 + 4 - \left(\frac{15\pi}{2} - 0 + 0\right) =$$

$$= \frac{20}{2}\pi + 4 - \frac{15}{2}\pi = \frac{5}{2}\pi + 4.$$

Problem 11. Sketch $r = \sin 6\theta$.



Problem 12. Find the area in one loop of $r = \sin 6\theta$.

$$A = \int \frac{1}{2} r^2(\theta)\, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2} \sin^2(6\theta)\, d\theta$$

$$\int_0^{\pi/6} \frac{1}{2} \frac{1}{2} \left(1 - \cos(12\theta)\right) d\theta$$

$$\frac{1}{4}\left[\theta - \frac{1}{12}\sin(12\theta)\right]_0^{\pi/6}$$

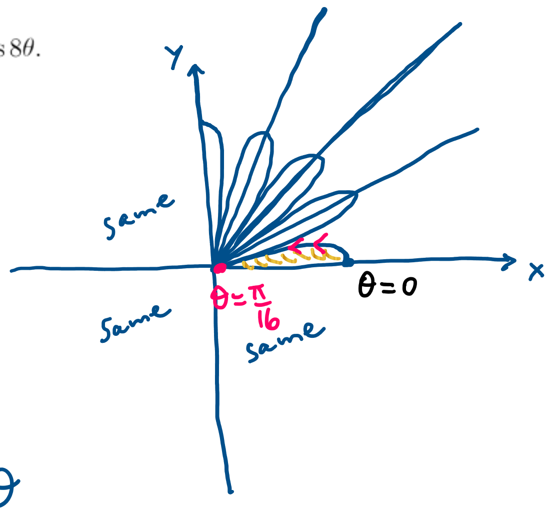
$$\frac{1}{4}\left[\frac{\pi}{6} - \frac{1}{12}\sin 2\pi - 0 + 0\right] = \frac{\pi}{24}.$$

Problem 13. Sketch $r = \cos 8\theta$.

$$\cos 8\theta = 0$$

$$8\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{16}$$



$$A = \cancel{2}\int_0^{\pi/16} \boxed{\frac{1}{\cancel{2}} \cos^2(8\theta)}\, d\theta$$

$$\int_0^{\pi/16} \frac{1}{2}\left(1 + \cos(16\theta)\right) d\theta =$$

$$= \frac{1}{2}\left[\theta + \frac{1}{16}\sin(16\theta)\right]_0^{\pi/16}$$

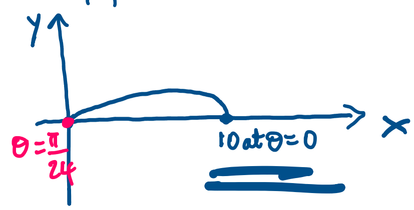
$$= \frac{1}{2}\left[\frac{\pi}{16} + 0\right] = \frac{\pi}{32}.$$

Problem 14. Find the area in one loop of $r = \cos 8\theta$.

Problem 15. Sketch $r = 10\cos 12\theta$.

24 petals.

$$\left.\begin{aligned} \cos(12\theta) &= 0 \\ 12\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{24} \end{aligned}\right\}$$



$$A = \cancel{2}\int_0^{\pi/24} \frac{1}{\cancel{2}}\, 100 \cos^2(12\theta)\, d\theta$$

$$= 100 \int_0^{\pi/24} \frac{1}{2}\left(1 + \cos(24\theta)\right) d\theta$$

$$= 100\, \frac{1}{2}\left[\theta + \frac{1}{24}\sin 24\theta\right]_0^{\pi/24} =$$

$$= \overset{25}{\cancel{50}}\left[\frac{\pi}{\underset{12}{\cancel{24}}} + 0 - 0 - 0\right] = \frac{25}{12}\pi$$

↑ done above

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Problem 16. Find the area in one loop of $r = 10\cos 12\theta$.

Done ✓