TEST REVIEW

Problem 1. Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{2ne^{-n}}{n^3+4n^2}$.

$$a_n = \frac{2n}{e^n(n^3+4n^2)} \approx \frac{2\cancel{n}}{e^n n^{\cancel{3}2}} \approx \frac{2}{n^2 e^n} \le \sum \frac{2}{e^n}$$

and $\frac{2}{n^2 e^n} \le \sum \frac{2}{n^2}$

p-series $p=2>1$

Limit Comp. Test

$$\frac{\dfrac{\cancel{2}n}{e^n(n^3+4n^2)}}{\dfrac{\cancel{2}}{n^2}} = \frac{n\,n^2}{e^n(n^3+4n^2)} \approx \frac{n^3}{e^n n^3} \to \frac{1}{e^n} \to 0$$

So our series converges by LCT.

Problem 2. Determine whether the series converges or diverges: $\sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{2+5\sqrt{n^3}}$.

$$a_n = \frac{\sqrt{n}-1}{2+5\sqrt{n^3}} \approx \frac{\sqrt{n}}{5\sqrt{n^3}} = \frac{n^{1/2}}{5\,n^{3/2}} = \frac{1}{5n}$$

$\sum \frac{1}{n}$ div.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}-1}{2+5\sqrt{n^3}}}{\frac{1}{n}} = \lim_{n} \frac{n\sqrt{n}-n}{2+5\sqrt{n^3}} = \lim_{n} \frac{n\,n^{1/2}}{5\,n^{3/2}} = \frac{1}{5}$$

So the given series diverges by LCT with the harmonic series $\sum \frac{1}{n}$.

(Can also do Ratio Test)

Problem 3. Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{(-5)^{n+1} n^4}{2^{2n}}$.

$$a_n = (-1)^{n+1} \frac{5\ 5^n\ n^4}{(2^2)^n} = (-1)^{n+1} 5 n^4 \left(\frac{5}{4}\right)^n$$

$\lim_n a_n$ DNE

$a_n \not\to 0$
so the
series
diverges.

Also divergent by Ratio Test.

$$\left| \frac{(-5)^{n+2} \cancel{(n+1)^4}}{2^{2n+2}} \cdot \frac{2^{2n}}{(-5)^{n+1} \cancel{n^4}} \right| \text{"}\to\text{"} \frac{5}{2^2} = \frac{5}{4} > 1 \text{ divergent}$$

Problem 4. Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{n(3+\ln n)^3}$.

$n(3+\ln n)^3 > n(\ln n)^3$

$$\sum \frac{1}{n(3+\ln n)^3} \leq \sum \frac{1}{n(\ln n)^3}$$ convergent

$$\int_1^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_e^{\infty} \frac{1}{u^3}\,du$$

$e^u = e^{\ln x} = x$

$u = \ln x$

$du = \frac{1}{x}$

$\int u^{-3} = \frac{u^{-2}}{-2}$

$$-\frac{1}{2}u^{-2}\Big|_e^{\infty} = -\frac{1}{2}\left(\frac{1}{\infty^2} - \frac{1}{e^2}\right) = \frac{1}{2e^2}.$$

The series converges by comparison, plus integral test.

Problem 5.
For what values of x does the series $\sum_{n=0}^{\infty} \frac{(x+3)^{n+1}}{5^n}$ converge? Find the sum of the series for those values of x.

Ratio Test

$$\lim_{n} \left| \frac{(x+3)^{\cancel{n+2}}}{5^{\cancel{n+1}}} \cdot \frac{\cancel{5^n}}{\cancel{(x+3)}^{n+1}} \right| = \lim_{n} \frac{|x+3|}{5} < 1$$

$$|x+3| < 5$$

$$-5 < x+3 < 5$$

$$-3 \qquad -3 \qquad -3$$

$$-8 < x < 2$$

$x=-8$ $\quad \sum \frac{(-5)^{n+1}}{5^n} = \sum (-1)^{n+1} 5$ diverg.

$x=2$ $\quad \sum \frac{5^{n+1}}{5^n} = \sum 5$ diverg.

Series converges for x in $(-8, 2)$.

$$\sum_{n=0}^{\infty} (x+3)\left(\frac{x+3}{5}\right)^n = (x+3) \underbrace{\sum_{0}^{\infty} \left(\frac{x+3}{5}\right)^n}_{\text{Geometric}}$$

$$(x+3) \frac{1}{1-\left(\frac{x+3}{5}\right)} = (x+3) \frac{5}{5-x-3}$$

$$\boxed{(x+3)\left(\frac{5}{2-x}\right)}.$$

Problem 6. Determine whether the series converges absolutely, conditionally, or diverges: $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{4^{n+1}}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{2n+2}}{4^{n+2}} \cdot \frac{4^{n+1}}{3^{2n}}\right| = \frac{3^2}{4} > 1$$

So the series does not converge absolutely.

~~Alt. Series Test~~ $\cancel{\frac{3^{2n}}{4^{n+1}}}$

$$\boxed{a_n = (-1)^n \frac{3^{2n}}{4^{n+1}}}$$

$\lim a_n$ dne so $a_n \not\to 0$ and the series diverges by Test for divergence.

Problem 7. Determine whether the series converges absolutely, conditionally, or diverges: $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{\sqrt{n}}$.

Absol. Conv.? $\sum_{1}^{\infty} \frac{e^{1/n}}{\sqrt{n}} \geq \sum_{1}^{\infty} \frac{1}{\sqrt{n}}$ p-series $p = \frac{1}{2} < 1$ so divergent

No absolute convergence

$\sum_{1}^{\infty} (-1)^n \frac{e^{1/n}}{\sqrt{n}}$ $\qquad a_{n+1} \leq a_n$

$f(x) = \frac{e^{1/x}}{\sqrt{x}} \qquad f'(x) = \frac{e^{1/x}\left(-\frac{1}{x^2}\right)\sqrt{x} - e^{1/x}\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} =$

$\frac{-\frac{e^{1/x}}{x^2} - \frac{e^{1/x}}{2\sqrt{x}}}{x}$ NEGATIVE so $f \searrow 0$

series converges by the AST.

The series is conditionally convergent.

Problem 8. Determine whether the series converges absolutely, conditionally, or diverges: $\sum_{n=1}^{\infty} \frac{\ln(n)(-2)^{n+1}}{3 \cdot n!}$

Ratio test

$$\lim_{n} \left| \frac{\ln(n+1)\,(-2)^{n+2}}{\cancel{3}\,(n+1)!} \cdot \frac{\cancel{3}\,n!}{\ln n\,(-2)^{n+1}} \right|$$

$$\lim_{n\to\infty} \frac{2}{n+1} \cdot \frac{\ln(n+1)}{\ln n} =$$

$$= \lim \frac{2}{n+1} = 0$$

$$\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \frac{\infty}{\infty}$$

$$\text{L'H.R. } \lim \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim \frac{x}{x+1} = 1$$

The series converges absolutely by the Ratio Test.

Problem 9. Determine whether the series converges absolutely, conditionally, or diverges: $\sum_{n=1}^{\infty} \frac{\ln(n)(-2)^{n+1}}{3 \cdot n!}$

done

$$\frac{a}{1-r} = \sum_{n=0}^{\infty} a r^n$$

Problem 10. Find a power series representation of $f(x) = \frac{1}{1-5x}$ and determine the radius and interval of convergence.

$$\frac{1}{1-5x} = \sum_{0}^{\infty} (5x)^n = \sum_{0}^{\infty} 5^n x^n$$

Geometric, ratio $= 5x$

$|5x| < 1$ so $\boxed{|x| < \frac{1}{5}} \Rightarrow R = \frac{1}{5}$

converges for x in $\left(-\frac{1}{5}, \frac{1}{5}\right)$

$x = -\frac{1}{5}$ $\quad \sum \left(-5\, \frac{1}{5}\right)^n = \sum (-1)^n$ div.

$x = \frac{1}{5}$ $\quad \sum \left(5\, \frac{1}{5}\right)^n = \sum 1^n$ div

$R = \frac{1}{5} \quad I = \left(-\frac{1}{5}, \frac{1}{5}\right).$

$$\frac{a}{1-r}$$

Problem 11. Find a power series representation of $f(x) = \dfrac{3}{2+x}$ and determine the radius and interval of convergence.

$$\frac{3}{2+x} = 3\left(\frac{1}{2+x}\right) = 3\,\frac{1}{2\left(1+\frac{x}{2}\right)} = \left(\frac{3}{2}\right)\frac{1}{1-\left(-\frac{x}{2}\right)}$$

Geom. series
ratio $= -\frac{x}{2}$

$$\frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\,3}{2^{n+1}}x^n$$

convergent for $\left|-\frac{x}{2}\right| < 1$ so $|x| < 2$

conv. in $(-2, 2)$ $\qquad \boxed{R = 2}$

$x = -2 \qquad \frac{3}{2}\sum\left(\frac{2}{2}\right)^n = \frac{3}{2}\sum 1$ DIV.

$x = 2 \qquad \frac{3}{2}\sum\left(\frac{-2}{2}\right)^n = \frac{3}{2}\sum(-1)^n$ DIV.

$I = (-2, 2)$.

Problem 12. Find a power series representation of $f(x) = \dfrac{x^4}{8 - x^3}$ and determine the radius and interval of convergence.

$$\frac{x^4}{8-x^3} = x^4 \, \frac{1}{8-x^3} = x^4 \, \frac{1}{8\left(1-\frac{x^3}{8}\right)} = \frac{x^4}{8} \, \frac{1}{1-\left(\frac{x}{2}\right)^3}$$

Geometric series

ratio $= \dfrac{x^3}{2^3} = \dfrac{x^3}{8}$

$$\frac{x^4}{8} \sum_{n=0}^{\infty} \left(\frac{x^3}{8}\right)^n = \frac{x^4}{8} \sum \frac{x^{3n}}{8^n} = \boxed{\sum_{n=0}^{\infty} \frac{x^{3n+4}}{8^{n+1}}}$$

$3(n+1)+4$

Ratio Test

$$\left| \frac{x^{3n+7}}{8^{n+2}} \cdot \frac{8^{n+1}}{x^{3n+4}} \right| = \left| \frac{x^3}{8} \right| < 1 \quad \text{so} \quad |x^3| < 8$$

$|x|^3 < 8$ so $|x| < 2$ $\qquad (-2, 2) \qquad R = 2$

$x = -2 \qquad \sum \dfrac{(-2)^{3n+4}}{2^{3n+3}} = \qquad (2^3)^{n+1} = 2^{3n+3}$

$= \sum (-1)^{3n+4} \cdot 2$ divergent

$x = 2 \qquad \sum \dfrac{2^{3n+4}}{2^{3n+3}} = \sum 2$ divergent

$I = (-2, 2), \quad R = 2.$

Problem 13. Find a power series representation of $f(x) = \dfrac{x^4}{8 - x^3}$ and determine the radius and interval of convergence.

done

$$\frac{\cancel{n} \cdot n \cdot n \cdot n}{\cancel{n}(n-1)(n-2)}$$

Problem 14. If $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{n^4(8x)^n}{n!}}$, find $f'(x)$ and $\int f(x)\,dx$.

$$f(x) = \sum_{n=1}^{\infty} \frac{n^3 \; 8^n}{(n-1)!}(x^n)$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n^3 \; 8^n}{(n-1)!}(n\, x^{n-1}) = \sum_{n=1} \frac{n^4\, 8^n}{(n-1)!} x^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{n^4\, 8^n}{n!} \int x^n\,dx =$$

$$= C + \sum_{n=0}^{\infty} \frac{n^4\, 8^n}{n!} \frac{x^{n+1}}{(n+1)} = C + \sum \frac{n^4\, 8^n}{(n+1)!} x^{n+1}.$$

Problem 15. Find a power series representation of $\frac{1}{(5+2x)^2}$ and determine the radius and interval of convergence.

$$\boxed{g(x) = \frac{1}{5+2x}} = (5+2x)^{-1} \Rightarrow g'(x) = -1(5+2x)^{-2}(2) =$$

$$g'(x) = \frac{-2}{(5+2x)^2}$$

$$\boxed{\frac{1}{-2}} g'(x) = \boxed{\frac{1}{(5+2x)^2}}$$

our series is $\left(-\frac{1}{2}\right) g'(x)$

$$g(x) = \frac{1}{5\left(1+\frac{2}{5}x\right)} = \frac{1}{5} \frac{1}{1-\left(-\frac{2}{5}x\right)}$$

Geom. ratio $-\frac{2}{5}x$

$$g(x) = \frac{1}{5} \sum_{n=0}^{\infty} \left(-\frac{2}{5}x\right)^n = \frac{1}{5} \sum \left(-\frac{2}{5}\right)^n x^n \Rightarrow$$

$$\Rightarrow g'(x) = \frac{1}{5} \sum_{0}^{\infty} \left(-\frac{2}{5}\right)^n n x^{n-1}$$

OUR SERIES is $-\frac{1}{2} \frac{1}{5} \sum_{0}^{\infty} \frac{(-2)^n}{5^n} n x^{n-1}$

$$\boxed{\sum_{0}^{\infty} (-1)^{n+1} \frac{2^{n-1} n}{\cancel{2} \cdot 5^{n+1}} x^{n-1}}$$

conv. when $\left|-\frac{2}{5}x\right| < 1$ so $|x| < \frac{5}{2}$

$R = 2.5$ $\qquad I = \left(-\frac{5}{2}, \frac{5}{2}\right)$.

Problem 16. Find a power series representation of $\frac{1}{(5+2x)^2}$ and determine the radius and interval of convergence.

Problem 17. Find a power series representation of $\frac{x^2}{(5+2x)^2}$ and determine the radius and interval of convergence.

multiply the series above by x^2

$$x^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{n-1} n}{5^{n+1}} x^{n-1}$$

$$\boxed{\sum_{0}^{\infty} (-1)^{n+1} \frac{2^{n-1} n}{5^{n+1}} x^{n+1}}$$

Problem 18. Find a power series representation of $x\ln(2-x^2)$ and determine the radius and interval of convergence.

$\left|\frac{x^2}{2}\right|<1$ so $x^2<2$

$(-\sqrt{2}, +\sqrt{2})$

$R=\sqrt{2}$

$$\int \frac{d}{dx} \ln(2-x^2) = \int \frac{-2x}{2-x^2}\,dx$$

$$-2x \frac{1}{2\left(1-\frac{x^2}{2}\right)} = -x \frac{1}{1-\frac{x^2}{2}}$$

Geom, ratio $\frac{x^2}{2}$

$$-x\sum_0 \left(\frac{x^2}{2}\right)^n = \sum_0^\infty - \frac{x^{2n-1}}{2^n}$$ now, integrate

$$C + \sum\left(-\frac{1}{2^n}\right)\int x^{2n-1}\,dx = C + \sum_{n=1}^{\infty}\left(-\frac{1}{2^n}\right)\frac{x^{2n}}{2n} = \ln(2-x^2)$$

plug in $x=0 \Rightarrow C=\ln 2$

$$x\left[\ln 2 + \sum_{1}^{\infty}\left(\frac{-1}{2^n\, 2n}\right)x^{2n}\right] =$$

$$= x\ln 2 + \sum_1^\infty \left(\underbrace{\frac{-1}{2^n(2n)}}_{2^{n+1}(n)} x^{2n+1}\right)$$

Problem 19. Find a power series representation of $\int x \ln(2 - x^2)\, dx$ and determine the radius and interval of convergence. same $R = \sqrt{2}$, $I = (-\sqrt{2}, \sqrt{2})$

$$\int x \ln 2 + \sum_{1}^{\infty} \frac{-1}{n\, 2^{n+1}} \; x^{2n+1} \quad dx \quad =$$

$$= C + \frac{x^2}{2} \ln 2 + \sum_{1}^{\infty} \frac{-1}{n\, 2^{n+1}} \; \frac{x^{2n+2}}{2n+2}$$

Problem 20. Express $\int_{x=0}^{1} e^{x^2} dx$ as a series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{0}^{\infty} \frac{x^{2n}}{n!}$$

$\int x^{2n}$

$$\int e^{x^2}\, dx = C + \sum_{0}^{\infty} \frac{1}{n!} \frac{x^{2n+1}}{2n+1}$$

$$\int_0^1 e^{x^2} dx = \sum_{0}^{\infty} \frac{1}{(2n+1)n!} - 0$$

Problem 21. Find $f^{(50)}(2)$ if $f(x) = \sum_{n=0}^{\infty} \frac{10^{n+1}(x-2)^n}{(n+5)!}$, that is the 50^{th} derivative of f at $x = 2$.

$c_n = \frac{10^{n+1}}{(n+5)!}$

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(2)}{n!}} (x-2)^n$$

$$c_n = \frac{f^{(n)}(2)}{n!} \Rightarrow f^{(n)}(2) = (n!)\, c_n$$

$$f^{(50)}(2) = 50!\, c_{50} = 50! \cdot \frac{10^{50+1}}{(50+5)!} = \boxed{\frac{10^{51}}{51 \cdot 52 \cdot 53 \cdot 54 \cdot 55}}$$

Problem 22. Find the Taylor Series centered at $a = 4$ if $f(x) = \dfrac{1}{x^2}$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n$$

$f^{(0)} = \frac{1}{x^2} = x^{-2}$

$f' = -2x^{-3}$

$f'' = (-2)(-3)x^{-4}$

$f''' = (-2)(-3)(-4)x^{-5}$

$\vdots$

$f^{(n)} = (-1)^n (n+1)!\, x^{-n-2}$

$f^{(n)}(4) = (-1)^n (n+1)!\, 4^{-n-2}$

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!\,4^{n+2}}(x-4)^n}$$

Problem 23. Find the Maclaurin Series for $f(x) = e^{-x^2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

Problem 24. Find the Maclaurin Series for $f(x) = x^2 \sin\left(\frac{x^4}{4}\right)$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$x^2 \sin\left(\frac{x^4}{4}\right) = x^2 \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{x^4}{4}\right)^{2n+1}$$

$4(2n+1)+2$

$8n+4+2$

$$\boxed{x \sin\left(\frac{x^4}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n+1}(2n+1)!} x^{8n+6}}$$

Problem 25. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n (\pi)^{2n}}{4^{2n}(2n)!}$. $= \frac{\sqrt{2}}{2}$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\quad x = \frac{\pi}{4}$

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Problem 26. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n}}{4^{2n}(2n)!}$.

$x=4$ center $= 0$

Problem 27. Assume that $\sum_{k=0}^{\infty} c_k 4^k$ converges. What can we say about:

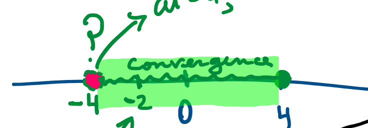
(1) $\sum_{k=0}^{\infty} c_k(-2)^k$.

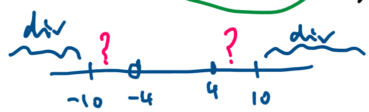
$\sum c_k x^k$

$x=-2$.

Converges

at -4, we don't know.





If $\sum c_k 10^k$ diverges

$\sum c_k (-12)^k$ diverges.

$\sum c_k (8)^k$ we don't know.

(2) $\sum_{k=0}^{\infty} c_k(-4)^k$. } we don't know.

$x=-4$ ...