Week 9

Monday, April 10, 2023 5:51 PM

151WIRH9

MATH 151- WEEK-IN-REVIEW 8

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END OF DERIVATIVES. BEGINNING OF ANTI-DERIVATIVES.

1. Find the given limits.

(a) $\lim_{x\to\infty}\left(\frac{2x^2}{2x+1}-\frac{x^2}{x+3}\right)$ $= \lim_{x\to\infty}\left(\frac{2x^2(x+3)-x^2(2x+1)}{(2x+1)(x+3)}\right)$

$\infty - \infty$

$= \lim_{x\to\infty}\left(\frac{x^2(2x+6-2x-1)}{2x^2+7x+3}\right)$

$= \lim_{x\to\infty}\left(\frac{5x^2}{2x^2+7x+3}\right) \overset{L'H}{=} \lim_{x\to\infty}\frac{10x}{4x+7} \overset{L'H}{=} \lim_{x\to\infty}\frac{10}{4}$

$= \frac{5}{2}$

(b) $\lim_{x\to 1}\left(\frac{e^{3x-3}+x^3-2}{5\ln(x)+4x-4}\right) = \frac{1+1-2}{0+4-4} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 1}\frac{3e^{3x-3}+3x^2}{\frac{5}{x}+4} = \frac{6}{9} = \boxed{\frac{2}{3}}$

(c) $\lim_{x\to\infty} (1 + x + x^2)^{\frac{1}{\ln(x)}}$ $\quad \infty^0$

$$\ln(A) = \lim_{x\to\infty} \ln\left((1+x+x^2)^{\frac{1}{\ln(x)}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{\ln(x)} \ln(1+x+x^2)\right)$$

$0 \quad \infty$

$$= \lim_{x\to\infty}\left(\frac{\ln(1+x+x^2)}{\ln(x)}\right) = \frac{\infty}{\infty} \qquad \frac{1}{1+x+x^2}\cdot(1+2x)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{1+2x}{1+x+x^2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty}\left(\frac{1+2x}{1+x+x^2}\right)\cdot\left(\frac{x}{1}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x+2x^2}{1+x+x^2}\right) = \lim_{x\to\infty}\left(\frac{2x^2}{x^2}\right) = 2 = \ln(A)$$

$$\boxed{A = e^2}$$

(d) $\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right)$

$\sin(0)=0$

$\infty \cdot 0$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{1/x} = \frac{0}{0} \qquad \pi x^{-1} \to -1\pi x^{-2}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}} \qquad \left(\cancel{-}\frac{\pi}{\cancel{x^2}}\right)\cdot\left(\cancel{-}\frac{\cancel{x^2}}{1}\right)$$

$$= \lim_{x\to\infty} \cos\left(\frac{\pi}{x}\right)(\pi) \qquad \frac{\pi}{\infty} = 0$$

$$= \lim_{x\to\infty} \cos(0)\cdot\pi = \boxed{\pi}$$

(e) $\lim_{x\to 0^+} x^3\ln(x)$ $= 0\cdot(-\infty)$

$= \lim_{x\to0^+} \frac{\ln(x)}{1/x^3}$ $\frac{-\infty}{\infty}$

$x^{-3} \to -3x^{-4}$

$\overset{L'H}{=} \lim_{x\to0^+} \frac{1/x}{-3/x^4}$

$= \lim_{x\to0^+}\left(-\frac{x^4}{3x}\right) = \lim_{x\to0}\left(-\frac{x^3}{3}\right)$

$= \boxed{0}$

(f) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{5x}$

$\ln(A) = \lim_{x\to\infty} \ln\left(\left(1+\frac{3}{x}\right)^{5x}\right)$

$= \lim_{x\to\infty} 5x \ln\left(1+\frac{3}{x}\right) = \infty\cdot 0$

$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{1/(5x)}$ $\frac{1}{5}\left(\frac{1}{x}\right)$

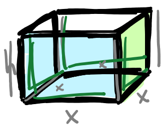
$\overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1+3/x}\right)\left(+3\cdot\cancel{\frac{1}{x^2}}\right)}{+\frac{1}{5}\cdot\cancel{\frac{1}{x^2}}}$

$= \lim_{x\to\infty}\left(\frac{3}{1+3/x}\right)\left(\frac{5}{1}\right)$

$\ln(A) = 15$

$\boxed{A = e^{15}}$

2. *(Spring 2012)* A rectangular basket with a square base is formed by putting decorative material over the bottom and sides of a metal frame. The frame is to be constructed by cutting a 70 foot bar of metal and assembling as in the figure below. Find the dimensions of the box which maximize the amount of material needed. Clearly show or explain why your answer is a maximum.



OPTIMIZATION

$$\text{Fabric} = \underbrace{x^2}_{\text{Bottom}} + \underbrace{4xh}_{\text{Sides}}$$

$$F = x^2 + \cancel{4}x\left(\frac{70-8x}{\cancel{4}}\right)$$

$$= x^2 + 70x - 8x^2$$

$$= 70x - 7x^2$$

$$F' = 70 - 14x$$

$$0 = 70 - 14x$$

$$x = \frac{70}{14} = 5\text{ ft}$$

CONSTRAINT

$$70 = 8x + 4h$$

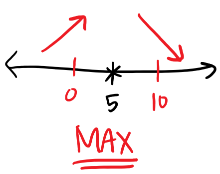
$$70 - 8x = 4h$$

$$h = \frac{70-8x}{4}$$

$$h = \frac{70-8(5)}{4}$$

$$= \frac{30}{4} = 7.5\text{ft}$$

FIRST DERIV. TEST

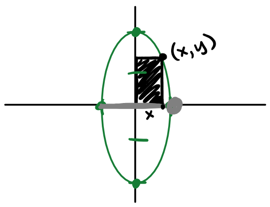


SECOND DERIVATIVE TEST

$$F'' = -14$$

Always concave down
$\Rightarrow$ Max

3. *(Spring 2013)* Find the base of the rectangle with largest area which can be inscribed in the first quadrant of the ellipse $\boxed{x^2 + \frac{y^2}{4} = 1}$. Clearly show that your answer yields maximum area.



$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

$\sqrt{1} = 1$
x-radius = 1

$\sqrt{4} = 2$ ← y-radius

OPTIMIZATION

$A = x \cdot y$

$A = x \cdot \sqrt{4-4x^2}$

$= x(4-4x^2)^{1/2}$

$A' = \underbrace{x \cdot \frac{1}{2}(4-4x^2)^{-1/2}(-8x)} + \underbrace{(4-4x^2)^{1/2}}$

$= (4-4x^2)^{-1/2}\left[\underbrace{-4x^2} + \underbrace{(4-4x^2)}\right]$

$= \boxed{\dfrac{4-8x^2}{(4-4x^2)^{1/2}}}$

CONSTRAINT

$x^2 + \frac{y^2}{4} = 1$

$\frac{y^2}{4} = 1 - x^2$

$y^2 = 4 - 4x^2$

$y = \sqrt{4-4x^2}$

$\frac{1}{2} < \frac{1}{1.7}$

$\rightarrow 4 - 8x^2 = 0$

$8x^2 = 4$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

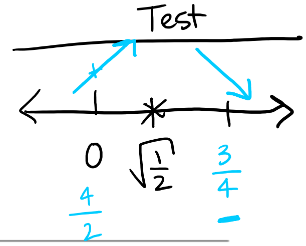
$x = \sqrt{\frac{1}{2}}$

$4 - 4x^2 = 0$

$x^2 = 1$

$\cancel{x = 1}$

First Derivative Test



MAX

$$\frac{4-8\left(\frac{3}{4}\right)^2}{\left(4-4\left(\frac{3}{4}\right)^2\right)^{1/2}}$$

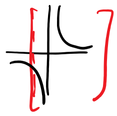
$$= \frac{4-\cancel{8}\left(\frac{9}{\cancel{16}\,2}\right)}{\left(4-4\left(\frac{9}{16}\right)\right)^{1/2}}$$

$$= \frac{4-\left(\frac{9}{2}\right)}{\left(4-\frac{9}{4}\right)^{1/2}} \quad \begin{matrix} <0 \\ >0 \end{matrix}$$

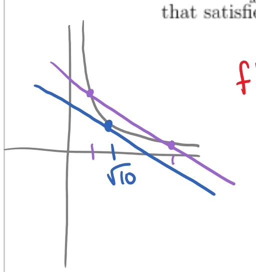
4. If $f(x) = \frac{1}{x}$, verify $f(x)$ satisfies the Mean Value Theorem on the interval $[-1, 10]$ and find all c that satisfies the conclusion of the Mean Value Theorem.

Not Continuous!

$\therefore$ MVT doesn't hold



5. If $f(x) = \frac{1}{x}$, verify $f(x)$ satisfies the Mean Value Theorem on the interval $[1, 10]$ and find all c that satisfies the conclusion of the Mean Value Theorem.



$f'(x) = -\frac{1}{x^2}$

$m = \frac{f(10) - f(1)}{10 - 1}$

$= \frac{\frac{1}{10} - 1}{10 - 1} = \frac{-\frac{9}{10}}{\frac{9}{1}} = -\frac{1}{10}$

$-\frac{1}{x^2} = -\frac{1}{10}$

$x^2 = 10$

$x = \boxed{\sqrt{10} = c}$

6. Find the antiderivatives of the following functions.

(a) $f(x) = \frac{5}{\sqrt{1 - x^2}} - \underbrace{\frac{7 + 3x - x^4}{x}}_{-\frac{7}{x} - 3 + x^3} + \underbrace{\frac{1}{1 + x^2}}$

$\frac{x^{3+1}}{3+1}$ $\cancel{\frac{x^{-1+1}}{0}}$

$F(x) = 5 \cdot \arcsin(x) - 7\ln|x| - 3x + \frac{x^4}{4} + \arctan(x) + C$

(b) $f(x) = 3x^2(x^3+1)$

$= 3x^5 + 3x^2$

$$F(x) = \boxed{\begin{aligned} &\frac{3x^6}{6} + \frac{3x^3}{3} + C \\ &= \frac{x^6}{2} + x^3 + C \end{aligned}}$$

(c) $f(x) = \dfrac{2x^2+6}{x^3} = \dfrac{2}{x} + \dfrac{6}{x^3}$ $\quad \dfrac{6x^{-3+1}}{-3+1}$ $\quad (2x^2+6)x^{-3}$

$$F(x) = \boxed{\begin{aligned} &2\ln|x| + \frac{6x^{-2}}{-2} + C \\ &= 2\ln|x| - \frac{3}{x^2} + C \end{aligned}}$$

(d) $f(x) = \csc(x)(\cot(x) - \csc(x))$

$= \csc(x)\cdot\cot(x) - \csc^2 x$

$$\boxed{F(x) = -\csc(x) + \cot(x) + C}$$

$\dfrac{1}{5}\dfrac{x^{-3+1}}{-3+1}$ $\quad \dfrac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1}$

$\dfrac{d}{dx}(7^x) = 7^x \cdot \ln(7)$

(e) $f(x) = 7^x + \dfrac{1}{5x^3} + \sqrt[5]{x^3}$ $\quad \dfrac{7^x}{\ln(7)}$

$$\boxed{\begin{aligned} F(x) &= \frac{7^x}{\ln(7)} + \frac{1}{5}\cdot\frac{x^{-2}}{-2} + \frac{x^{8/5}}{8/5} + C \\ &= \frac{7^x}{\ln(7)} - \frac{1}{10x^2} + \frac{5}{8}x^{8/5} + C \end{aligned}}$$

(f) $f'(x) = \underbrace{2(1-x^2)^{-1/2}}_{\frac{2}{\sqrt{1-x^2}}} + e^x$ with $f(0) = 4$.

$f(x) = 2\arcsin(x) + e^x + C$

$f(0) = 2\arcsin(0) + e^0 + C = 4$

$= 0 + 1 + C = 4$

$C = 3$

$\boxed{f(x) = 2\arcsin(x) + e^x + 3}$

(g) $f'(x) = 2e^x - 5$ with $f(0) = 1$.

$f(x) = 2e^x - 5x + C$

$f(0) = 2 - 0 + C = 1$

$C = -1$

$\boxed{f(x) = 2e^x - 5x - 1}$

(h) $f''(x) = 20x^3 + 6e^x$ with $f(0) = 4$ and $f(1) = 2$

$f'(x) = 5x^4 + 6e^x + C$

$f(x) = x^5 + 6e^x + Cx + D$

$f(0) = 0 + 6 + 0 + D = 4$

$D = -2$

$f(1) = 1 + 6e + C + D = 2$

$1 + 6e + C - 2 = 2$

$6e + C = 3$

$C = 3 - 6e$

$\boxed{f(x) = x^5 + 6e^x + (3-6e)x - 2}$