MATH 151 - WEEK-IN-REVIEW 6

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MORE DERIVATIVES

1. Differentiate the following functions. You don't need to simplify.

(a) $r(x) = \arctan(3x^2 - 1)$

$$r'(x) = \frac{1}{(3x^2-1)^2+1} \cdot 6x = \frac{6x}{(3x^2-1)^2+1}$$

(b) $r(x) = \arcsin(x^3 e^x)$

$$r'(x) = \frac{1}{\sqrt{1-(x^3e^x)^2}} \left[x^3e^x + e^x \cdot 3x^2\right]$$

$$= \frac{x^3e^x + e^x \cdot 3x^2}{\sqrt{1-x^6e^{2x}}}$$

(c) $f(t) = \ln(4t - 6t^2)$

$$f'(t) = \frac{1}{4t-6t^2}(4-12t)$$

$$= \frac{4-12t}{4t-6t^2}$$

(d) $g(x) = \cos(\log_4(x))$

$$g'(x) = -\sin(\log_4(x)) \cdot \frac{1}{x\ln(4)}$$

(e) $y = \left(\ln(3x)\right)^{\csc(x)}$

$\ln y = \ln\left(\ln(3x)\right)^{\csc x}$

Rewrite $\ln y = \csc(x) \cdot \ln(\ln(3x))$

$\frac{1}{y} y' = \left[\csc(x) \cdot \frac{1}{\ln(3x)} \cdot \frac{1}{3x} \cdot 3 + \ln(\ln(3x)) \cdot (-\csc(x)\cot(x))\right] y$

$y' = \left[\csc(x) \cdot \frac{1}{\ln(3x)} \cdot \frac{1}{3x} \cdot 3 + \ln(\ln(3x)) \cdot (-\csc(x) \cdot \cot(x))\right] \left(\ln(3x)\right)^{\csc(x)}$

(f) $g(x) = \frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}$

$\ln(g(x)) = \ln\left(\frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}\right)$

Rewrite $\ln(g(x)) = 5\ln(4x+1) + 2\ln(6-5x) - \ln(2x^9) - \cancel{\ln}(\cancel{e}^{4x^2+7x})$

$\frac{1}{g(x)} \cdot g'(x) = 5 \cdot \frac{4}{4x+1} + 2 \cdot \frac{-5}{6-5x} - \frac{18x^8}{2x^9} - (8x+7)$

$$\boxed{g'(x) = \left(\frac{20}{4x+1} - \frac{10}{6-5x} - \frac{9}{x} - 8x - 7\right)\left(\frac{(4x+1)^5(6-5x)^2}{2x^9 e^{4x^2+7x}}\right)}$$

2. Find an equation of tangent line to the curve $y = 5x^3 \ln(x)$ at the point $(1, 0)$.

Point

$(1,0)$

$$\boxed{\begin{gathered} y - 0 = 5(x-1) \\ y = 5x - 5 \end{gathered}}$$

Slope

$y' = 5x^3 \cdot \frac{1}{x} + \ln(x) \cdot 15x^2$

$y'(1) = 5 \cdot 1 + \cancel{\ln(1) \cdot 15}$

$= 5$

$g'f + f'g = gf' + fg'$

3. Given $\mathbf{r}(t) = \langle 2\sin(t) + 2\cos(t), 3\cos(t) - 3\sin(t)\rangle$

(a) Find $\mathbf{r}'\left(\frac{2\pi}{3}\right)$.

vertical   horizontal

$$r'(t) = \langle 2\cos(t) - 2\sin(t), -3\sin(t) - 3\cos(t)\rangle$$

$$r'\left(\tfrac{2\pi}{3}\right) = \left\langle -1 - \sqrt{3}, -\frac{3\sqrt{3}}{2} + \frac{3}{2}\right\rangle$$

(b) Write the equation of the tangent line at $t = 0$.

Slope

$$r'(0) = \langle 2\cos(0) - 2\sin(0), -3\sin(0) - 3\cos(0)\rangle$$

$$= \underbrace{\langle 2, -3\rangle}_{m = -\frac{3}{2}}$$

point

$$r(0) = \langle 0 + 2, 3 - 0\rangle$$

$$= \langle 2, 3\rangle$$

point-slope form: $y - 3 = -\frac{3}{2}(x - 2)$

parametric form: $\vec{s}(t) = \langle 2t + 2, -3t + 3\rangle$

(c) Find the horizontal tangent line(s) for $\mathbf{r}(t)$.

$$-3\sin(t) - 3\cos(t) = 0$$

$$-3(\sin(t) + \cos(t)) = 0$$

$$\sin(t) + \cos(t) = 0$$

$$\frac{\sin(t)}{\cos(t)} = \frac{-\cos(t)}{\cos(t)}$$

$$\boxed{\tan(t) = -1 \quad \boxed{t = 3\pi/4, 7\pi/4}}$$

Losing an answer?
$\cos(t) = 0$
$t = \frac{\pi}{2}$

$$y\left(\tfrac{3\pi}{4}\right) = 3\cos\left(\tfrac{3\pi}{4}\right) - 3\sin\left(\tfrac{3\pi}{4}\right) = -3\sqrt{2}$$

$$y\left(\tfrac{7\pi}{4}\right) = 3\cos\left(\tfrac{7\pi}{4}\right) - 3\sin\left(\tfrac{7\pi}{4}\right) = 0$$

(d) Find the vertical tangent line(s) for $\mathbf{r}(t)$.

$$2\cos(t) - 2\sin(t) = 0$$

$$\cos(t) - \sin(t) = 0$$

$$\cos(t) = \sin(t)$$

$$1 = \tan(t)$$

$$t = \pi/4, 5\pi/4$$

$$x(\pi/4) = 2\left(\tfrac{\sqrt{2}}{2}\right) + 2\left(\tfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$x(5\pi/4) = 2\left(-\tfrac{\sqrt{2}}{2}\right) + 2\left(-\tfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}$$

4. Given $\mathbf{r}(t) = \langle t^4 - 24t + 5, 10t^5 + 1 \rangle$

(a) Find $\mathbf{r}'(1)$.

$$r'(t) = \langle 4t^3 - 24, 50t^4 \rangle$$

$$r'(1) = \langle -20, 50 \rangle$$

(b) Write the equation of the tangent line at $t = 0$.

Slope: $m_{tan} = \langle -24, 0 \rangle$ $\quad m = \frac{0}{-24} = 0$

point: $r(0) = \langle 5, 1 \rangle$

$$y - 1 = 0(x - 5)$$

$$\boxed{y = 1}$$

Parametric: $\boxed{\langle -24t + 5, 1 \rangle}$

(c) Find the horizontal tangent line(s) for $\mathbf{r}(t)$.

$$50t^4 = 0$$

$$t^4 = 0$$

$$t = 0$$

$$\boxed{y = 1}$$

(d) Find the vertical tangent line(s) for $\mathbf{r}(t)$.

$$0 = 4t^3 - 24$$

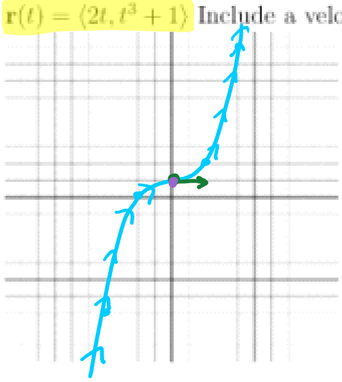
$$t^3 = 6$$

$$t = \sqrt[3]{6}$$

$$\boxed{x(\sqrt[3]{6}) = (\sqrt[3]{6})^4 - 24(\sqrt[3]{6}) + 5}$$

5. Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

(a) $\mathbf{r}(t) = \langle 2t, t^3 + 1 \rangle$ Include a velocity and acceleration vector for $t = 0$



t	x	y
−2	−4	−7
−1	−2	0
0	0	1
1	2	2
2	4	9

$r'(t) = \langle 2, 3t^2 \rangle = v(t)$

$r'(0) = \langle 2, 0 \rangle$

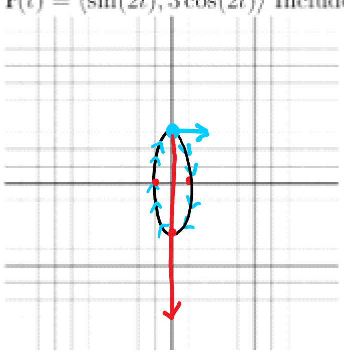
(tangent vector & flow direction)

$a(t) = \langle 0, 6t \rangle$

$a(0) = \langle 0, 0 \rangle$

L & R   Up & Down

(b) $\mathbf{r}(t) = \langle \sin(2t), 3\cos(2t) \rangle$ Include the velocity and acceleration vectors for $\boxed{t = 0}$



$(x)^2 = (\sin(2t))^2$ $y = 3\cos(2t)$

$\frac{y}{3} = \cos(2t)$

$\sin^2(2t) + \cos^2(2t) = 1$

$x^2 + \frac{y^2}{9} = 1$

at $t=0$ $r(0) = \langle \sin(0), 3\cos(0) \rangle = \langle 0, 3 \rangle$

$r'(t) = \langle 2\cos(2t), -6\sin(2t) \rangle$

$r'(0) = \langle 2, 0 \rangle$

$a(t) = r''(t) = \langle -4\sin(2t), -12\cos(2t) \rangle$

$a(0) = \langle 0, -12 \rangle$

6. At what point(s) on the curve $x = t^3 - t^2 - 14t$, $y = \frac{1}{2}t^2 - t$ is the tangent line parallel to the line with equations $x = 4t$, $y = 1 - 6t$?

$m = -\frac{6}{4} = -\frac{3}{2}$

$x'(t) = 3t^2 - 2t - 14$

$y'(t) = t - 1$

$\frac{dy}{dx} = \frac{t-1}{3t^2-2t-14} = -\frac{3}{2}$

$t - 1 = -3$

$\boxed{t=-2}$

$3t^2 - 2t - 14 = 2$

$3t^2 - 2t - 16 = 0$

$(3t-8)(t+2) = 0$

$\cancel{t = \frac{8}{3}}$ $\boxed{t=-2}$

Point: $x(-2) = -8 - 4 + 28 = -4$

$y(-2) = \frac{1}{2}(4) + 2 = 4$

$\boxed{(-4,4)}$

7. Find the angle between the velocity vector and the acceleration vector for $\mathbf{r}(t) = \langle t, 2t^3 \rangle$ at the point where t = 1.

$\vec{v}(t) = \langle 1, 6t^2 \rangle$

$\vec{v}(1) = \langle 1, 6 \rangle$

$\vec{a}(t) = \langle 0, 12t \rangle$

$\vec{a}(1) = \langle 0, 12 \rangle$

$\cos\theta = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\| \cdot \|\vec{a}\|} = \frac{72}{\sqrt{37} \cdot 12}$

$\boxed{\theta = \cos^{-1}\left(\frac{72}{\sqrt{37} \cdot 12}\right) = \cos^{-1}\left(\frac{6}{\sqrt{37}}\right)}$

8. A ball is thrown vertically upward with a velocity of 32 feet per second. The height after t seconds is given by $h(t) = 32t - 16t^2$. With what velocity does the ball hit the ground?

① When does it hit the ground?

$0 = 32t - 16t^2$

$0 = 16t(2-t)$

$t = 0$ $t = 2$

② $v(t) = h'(t) = 32 - 32t$

$h'(2) = 32 - 64$

$\boxed{= -32 \text{ ft/sec}}$

9. A particle moves according to the equation of motion $s(t) = 2t^3 - 6t^2 - 5$, where $s(t)$ is measured in meters and t in seconds.

(a) When is the particle at rest?

$s'(t) = 6t^2 - 12t = v(t)$

$0 = 6t(t-2)$

$\boxed{t=0 \quad t=2 \text{ seconds}}$

position
→ velocity ←
acceleration

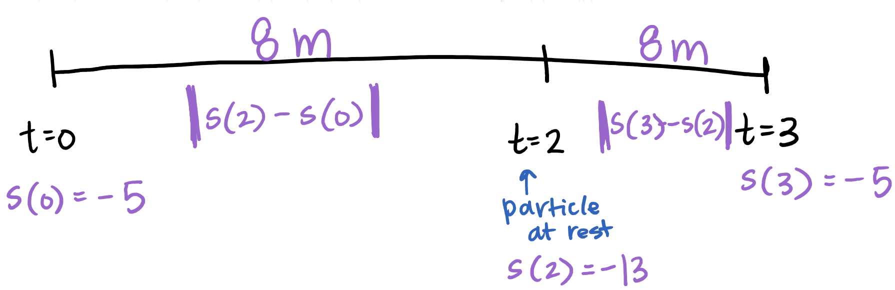
(b) What is the acceleration when the particle is at rest?

$a(t) = 12t - 12$

@ $t=0$ $a(0) = 0 - 12$
$= -12\ m/s^2$

@ $t=2$ $a(0) = 24 - 12$
$= 12\ m/s^2$

(c) What is the total distance traveled in the first 3 seconds?



$\boxed{16m}$

(d) What is the total displacement in the first 3 seconds?

$s(3) - s(0) = -5 - (-5) = \boxed{0\ m}$

10. During lab, I forgot to measure how much bacteria I started with, but after one hour there were 1000 bacteria. After five total hours, the number of bacteria has increased to 3500 bacteria. Find a formula for the number of bacteria after t hours. Find the number of bacteria and the rate of growth of the bacteria after 2 hours.

start $\to (1, 1000)$

future $\to (5, 3500)$

$y = Pe^{rt}$

$3500 = 1000e^{r4}$

$3.5 = e^{4r}$

$\ln(3.5) = 4r$

$r = \boxed{\frac{\ln(3.5)}{4}}$ rate of growth

time since we watched

$\boxed{y = 1000\, e^{\frac{\ln(3.5)}{4}(t-1)}}$

11. A particular drug has half life of 15 hours. If we begin with a sample size of mass 500 mg, how long will it take for this sample to decay to a mass of 125 mg?

$\boxed{500} \xrightarrow{15\text{ hrs}} \boxed{250} \xrightarrow{15\text{ hrs}} \boxed{125}$

$\boxed{30\text{ hrs}}$

12. A pie is taken from an oven where the temperature has reached 375° F and is placed on a counter in a room where the temperature is 75° F. If the temperature of the pie is 175° F after 30 minutes, when will the pie have cooled to 90° F?

$y = Pe^{rt} + A$

difference between object & room

temp of room

$175 = 300e^{r\cdot\frac{1}{2}} + 75$

$100 = 300e^{r\cdot\frac{1}{2}}$

$\frac{1}{3} = e^{r\cdot\frac{1}{2}}$

$\ln\left(\frac{1}{3}\right) = \frac{1}{2}r$

$r = 2\ln\left(\frac{1}{3}\right)$

$90 = 300e^{2\ln(\frac{1}{3})t} + 75$

$15 = 300e^{2\ln(\frac{1}{3})t}$

$\frac{1}{20} = e^{2\ln(\frac{1}{3})t}$

$\ln\left(\frac{1}{20}\right) = 2\ln\left(\frac{1}{3}\right)t$

$\boxed{t = \frac{\ln(\frac{1}{20})}{2\ln(\frac{1}{3})} = \frac{\ln(20)}{2\ln(3)}}$