Week 5

151WIRH5

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MATH 151- WEEK-IN-REVIEW 5

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DERIVATIVES

Basic Derivative Rules

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln(a)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Inverse Trig Derivatives

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$

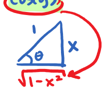
$\sin^{-1}(x) = y$

$x = \sin(y)$

$\downarrow$

$1 = \cos(y)\cdot y'$

$\frac{1}{\cos(y)} = y'$



Extra Derivative Rules

Product Rule

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\cdot g'(x)$$

1. Differentiate the following functions. You don't need to simplify.

(a) $f(x) = x^7 + \sqrt[4]{x} - \dfrac{5}{x} + \tan(x) - \csc(x)$

$x^{1/4}$ $\quad -5x^{-1}$

$f'(x) = 7x^6 + \frac{1}{4}x^{-3/4} + 5x^{-2} + \sec^2(x) + \csc x \cot x$

(b) $g(t) = (2t + 5)(3 - t)$ $\quad = 6t - 2t^2 + 15 - 5t$

$= -2t^2 + t + 15$

Product Rule

$f(t) = 2t+5 \quad h(t) = 3-t$

$f'(t) = 2 \quad h'(t) = -1$

$f \cdot h' + h \cdot f'$

$\boxed{g'(t) = (2t+5)(-1) + (3-t)(2)}$

Foil & Power Rule

$\boxed{g'(t) = -4t + 1}$

(c) $G(x) = \left(\dfrac{1}{x^2} - \dfrac{7}{x^5}\right)(3x - 2)$

$= (x^{-2} - 7x^{-5})(3x^1 - 2)$

Product Rule

$f(x) = x^{-2} - 7x^{-5} \quad g(x) = 3x-2$

$f'(x) = -2x^{-3} + 35x^{-6} \quad g'(x) = 3$

$\boxed{G'(x) = (x^{-2} - 7x^{-5})(3) + (3x-2)(-2x^{-3} + 35x^{-6})}$

$= 3x^{-1} - 2x^{-2} - 21x^{-4} + 14x^{-5}$

$\boxed{G'(x) = -3x^{-2} + 4x^{-3} + 84x^{-5} - 70x^{-6}}$

(d) $h(y) = \dfrac{7}{y^4}$ $= 7y^{-4}$

$$\boxed{h'(y) = -28y^{-5} = \frac{-28}{y^5}}$$

(e) $y = \dfrac{\sqrt[4]{x^3} + x}{x^2}$ $= \dfrac{x^{3/4}}{x^2} + \dfrac{x^1}{x^2} = x^{-5/4} + x^{-1}$

$f(x) = \sqrt[4]{x^3} + x$

$= x^{3/4} + x$

Quotient Rule

$y' = \dfrac{\text{Lo dHi} - \text{Hi dLo}}{\text{Lo}^2}$

$$= \boxed{\frac{x^2\left(\frac{3}{4}x^{-1/4} + 1\right) - \left(\sqrt[4]{x^3} + x\right)(2x)}{(x^2)^2}} \leftrightarrow \boxed{y' = -\frac{5}{4}x^{-9/4} - x^{-2}}$$

(f) $F(x) = \dfrac{x^2 - x}{4x + 5}$

$$\boxed{F'(x) = \frac{(4x+5)(2x-1) - (x^2 - x)(4)}{(4x+5)^2}}$$

(g) $q(x) = (2 - 3x + 5x^2)^{50}$

$$\boxed{q'(x) = 50\,(2 - 3x + 5x^2)^{49} \cdot (-3 + 10x)}$$

(h) $H(x) = \dfrac{x}{(x^3-7)^5}$ OR $H(x) = x(x^3-7)^{-5}$

$$\boxed{H'(x) = \frac{(x^3-7)^5(1) - x\cdot 5(x^3-7)^4\cdot(3x^2)}{(x^3-7)^{10}}}$$

(i) $P(x) = e^{\tan(x)} - 3^{5x^2-1}$

$y = b^{f(x)} \Rightarrow y' = b^{f(x)}\cdot f'(x)\cdot \ln(b)$

$$\boxed{P'(x) = e^{\tan(x)}\cdot \sec^2 x - 3^{5x^2-1}\cdot(10x)\cdot \ln(3)}$$

$y = 2^x$
$= e^{\ln(2^x)}$
$= e^{x\ln(2)}$
$y' = e^{x\ln(2)}\cdot \ln(2)$
$= 2^x\cdot \ln(2)$

(j) $f(x) = \ln(3x^2+5x-1) + \log_4(3-x)$

$$f'(x) = \frac{1}{3x^2+5x-1}(6x+5) + \frac{1}{(3-x)\ln(4)}(-1)$$

$$= \boxed{\frac{6x+5}{3x^2+5x-1} + \frac{-1}{(3-x)\ln(4)}}$$

(k) $y = e^{t\sin^2(t)}$

$$\boxed{y' = e^{t\sin^2 t}\cdot\left(t\cdot 2(\sin(t))\cos(t) + \sin^2(t)\right)}$$

Exponent

$t\sin^2(t) = \overset{f}{t}\,\overbrace{(\sin(t))^2}^{g}$

Derivative of exponent

$t\cdot 2(\sin(t))'\cdot\cos(t) + \sin^2 t\cdot 1$

(product rule)

(l) $r(x) = \arccos(7x+2)$

$$\boxed{r'(x) = \frac{-1}{\sqrt{1-(7x+2)^2}}\cdot 7}$$

$y = \arcsin(x)$

$y' = \dfrac{1}{\sqrt{1-x^2}}$

(m) $g(y) = \csc(\tan(\cos(y)))$

$$\boxed{g'(y) = -\csc(\tan(\cos(y)))\cot(\tan(\cos(y)))\cdot \sec^2(\cos(y))\,(-\sin(y))}$$

2. Find the 2023rd derivative of $y = 5\sin(5x)$

R1 $y' = 5\cos(5x)\cdot 5 = 25\cos(5x) = 5^2\cos(5x)$

R2 $y'' = -25\sin(5x)\cdot 5 = -125\sin(5x) = -5^3\sin(5x)$

R3 $y''' = -5^3\cos(5x)\cdot 5 = -5^4\cos(5x)$

$y^{(4)} = 5^5\sin(5x)$

$$\boxed{y^{(2023)} = -5^{2024}\cos(5x)}$$

$$\begin{array}{r} 505 \text{ R3} \\ 4\overline{)2023} \\ -20 \\ \hline 02 \\ -0 \\ \hline 23 \\ -20 \\ \hline 3 \end{array}$$

3. Find the 2023rd derivative of $y = xe^{2x}$

$f \quad g$

$f = x \qquad g = e^{2x}$

$f' = 1 \qquad g' = e^{2x}\cdot 2$

$y' = xe^{2x}\cdot 2 + e^{2x}$

$= e^{2x}(2x + 1)$

$f = e^{2x} \qquad g = 2x+1$

$f' = e^{2x}\cdot 2 \qquad g' = 2$

$y'' = e^{2x}\cdot 2 + e^{2x}\cdot 2(2x+1)$

$= 2e^{2x}(1 + (2x+1))$

$= 2e^{2x}(2x+2) = 4e^{2x}(x+1)$

$$\boxed{y^{(2023)} = 2^{2022}e^{2x}[2x + 2023]}$$

$y''' = 4e^{2x}(1) + (x+1)4e^{2x}\cdot 2$

$= 4e^{2x}(1 + (x+1)(2))$

$= 4e^{2x}(2x+3)$

2^2

$y^{(4)} = 4e^{2x}(2) + (2x+3)4e^{2x}\cdot 2$

$= 4e^{2x}(2)[1 + (2x+3)]$

$= 8e^{2x}[2x+4]$

2^3

$y = \frac{1}{x} = x^{-1}$

$y' = -x^{-2}$

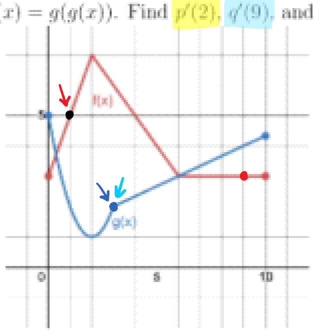
$y'' = 2x^{-3}$

$y''' = -2(3)x^{-}$

$y^{(4)} = 2(3)(4)$

$y^{(10)} = 10!\,x$

4. If f and g are the functions whose graphs are shown, let $p(x) = f(g(x))$, $q(x) = g(f(x))$, and $r(x) = g(g(x))$. Find $p'(2)$, $q'(9)$, and $r'(8)$.



$p'(x) = f'(g(x))\cdot g'(x)$

$p'(2) = f'(g(2))\cdot g'(2)$

$= f'(1)\cdot 0$

$= 2\cdot 0 = 0$

$q'(x) = g'(f(x))\cdot f'(x)$

$q'(9) = g'(3)\cdot 0$

DNE

5. Find $\frac{dy}{dx}$ for each of the following.

(a) $5x^2 - x^3y + 5y = 7$

Implicit Differentiation

$10x - \left(x^3\cdot \frac{dy}{dx} + y\cdot 3x^2\right) + 5\frac{dy}{dx} = 0$

$10x - x^3\frac{dy}{dx} - 3x^2y + 5\frac{dy}{dx} = 0$

$-x^3\frac{dy}{dx} + 5\frac{dy}{dx} = -10x + 3x^2y$

$\frac{dy}{dx}(-x^3+5) = -10x + 3x^2y$

$\boxed{\frac{dy}{dx} = \frac{-10x+3x^2y}{-x^3+5}}$

(b) $\sec(xy) = 3 - \cos(y)$

$\sec(xy)\tan(xy)\cdot\left[x\cdot\frac{dy}{dx} + y\right] = \sin(y)\cdot\frac{dy}{dx}$

$\sec(xy)\tan(xy)\cdot x\frac{dy}{dx} + \sec(xy)\tan(xy)\cdot y = \sin(y)\cdot\frac{dy}{dx}$

$\sec(xy)\tan(xy)\cdot y = \left[\sin(y) - \sec(xy)\tan(xy)\right]\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{\sec(xy)\tan(xy)\cdot y}{\sin(y) - \sec(xy)\tan(xy)}}$

CAN'T CANCEL!
The sin(y) in denom ruins it for us.

6. Find the equation of the tangent line to $y = 3x + \sqrt{x}$ at $\boxed{(4, 14)}$

$= 3x + x^{1/2}$

$y' = 3 + \frac{1}{2}x^{-1/2}$

$= 3 + \frac{1}{2\sqrt{x}}$

$y'(4) = 3 + \frac{1}{2\sqrt{4}} = 3 + \frac{1}{4} = \frac{13}{4}$

$\boxed{y - 14 = \frac{13}{4}(x - 4)}$

7. Find the equation of the tangent line to $\underbrace{x^2 + 6y^2 = 25}_{\text{Implicit Differentiation}}$ at $(1, 2)$.

$2x + 12y \cdot y' = 0$

$y' = \frac{-2x}{12y} = \frac{-x}{6y}$

$m = \frac{-1}{12}$

$\boxed{y - 2 = -\frac{1}{12}(x - 1)}$

8. Given $f(x) = \begin{cases} mx - b & \text{if } x < -1 \\ 5x^2 & \text{if } x \geq -1 \end{cases}$. Find values for m and b that make the function differentiable everywhere.

$f'(x) = \begin{cases} m & \text{if } x < -1 \\ 10x & \text{if } x \geq -1 \end{cases}$

f(x) continuous

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (mx - b) = -m - b$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (5x^2) = 5$

$-m - b = 5$

$10 - b = 5$

$\boxed{b = 5}$

f'(x) continuous

$\lim_{x \to -1^-} f'(x) = m$

$\lim_{x \to -1^+} f'(x) = \lim_{x \to -1^+} (10x) = -10$

$\boxed{m = -10}$

9. For what values of a and b [redacted] is the line $y = 3x - 1$ tangent to the parabola [redacted] when $x = 4$?

$y = ax^2 + b$

If the line is tangent, we need point AND slope to be the same.

Slope

Slope of line = 3

Slope of tangent line
$= 2ax$

Plug in 4: $3 = 8a \Rightarrow \boxed{a = \frac{3}{8}}$

Point

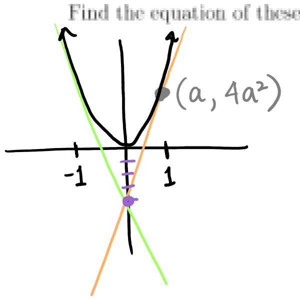
$y = \frac{3}{8}x^2 + b$

y-value on line:
$y = 12 - 1 = 11$

y-value on parabola:
$y = \frac{3}{8}(4)^2 + b = 6 + b$

Set = : $11 = 6 + b \Rightarrow \boxed{b = 5}$

10. Show there are two tangent lines to the parabola $y = 4x^2$ that pass through the point $(0, -4)$. Find the equation of these tangent lines.



Slope

$y' = 8x$

Point

$(0, -4)$

$(a, 4a^2)$

$y_2 - y_1 = m(x_2 - x_1)$

$4a^2 + 4 = 8a(a - 0)$

$4a^2 + 4 = 8a^2$

$4 = 4a^2$

$a^2 = 1$

$a = \pm 1$

At $x = 1$: $y + 4 = 8(x - 0)$

At $x = -1$: $y + 4 = -8(x - 0)$