TEXAS A&M UNIVERSITY
Math Learning Center

Math 151 - Fall 2022
"Hands on, grades up"

Week-in-Review 9: Exam 2 review
(Ch 3.1 - 3.10, K1, K2: Derivatives and their Applications.)

Problem 1. Find the following derivatives:

a) $y = \sin^5(\sec(\sqrt{x^2+1}))$

$$y' = \left[5\sin^4(\sec(\sqrt{x^2+1}))\right]\cdot\left[\cos(\sec(\sqrt{x^2+1}))\right]\cdot\left[\sec(\sqrt{x^2+1})\tan(\sqrt{x^2+1})\right]\cdot\left[\frac{1}{2\sqrt{x^2+1}}\right]\cdot[2x]$$

b) $f(x) = \underbrace{x^2}\,\underbrace{\arcsin(x)}$ → product rule

$$f'(x) = (2x)(\arcsin(x)) + (x^2)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

c) $f(x) = 6^{2^x} + \sqrt[3]{3x^2}$ → exponential → power fn

b=6

$$g(x) = 6^{2^x} = 6^{h(x)}$$
$$g'(x) = 6^{h(x)}\cdot\ln 6\cdot h'(x)$$
$$= 6^{2^x}\cdot\ln(6)\cdot 2^x\cdot\ln(2)$$

$$j(x) = \sqrt[3]{3x^2} = (3x^2)^{1/3}$$
$$= 3^{1/3}\cdot x^{2/3}$$
$$j'(x) = \underbrace{(3^{1/3})\frac{2}{3}}\cdot x^{-1/3}$$
$$2\cdot 3^{1/3-1}$$
$$= 2\cdot 3^{-2/3}$$

$$f'(x) = 6^{2^x}\cdot\ln(6)\cdot 2^x\cdot\ln(2) + \sqrt[3]{3}\cdot\frac{2}{3}\frac{1}{\sqrt[3]{x}}$$

chain Rule is ok.

$$\frac{d}{dx}(3x^2)^{1/3} = \frac{1}{3}(3x^2)^{-2/3}\cdot(6x)$$
$$= 2\cdot 3^{-2/3}\cdot x^{-\frac{4}{3}+1}$$
$$= 2\,3^{-2/3}\cdot x^{-1/3}$$

$= 2\,3^{\cdot\cdot} . x^{\cdot}$

2

d) $y = \ln(\underbrace{x^2 e^{-3x}})$ $= \ln(x^2) + \ln(e^{-3x})$

$= 2\ln(x) - 3x\ln(e)$

$= 2\ln(x) - 3x$

$\frac{dy}{dx} = 2 . \frac{1}{x} - 3 = \frac{2}{x} - 3$

e) $f(x) = \underbrace{x^2}\underbrace{\ln(3+2x)}$ → $\ln(A+B)$ → cannot be simplified further

→ product rule

ln → base e

$\ln(e) = 1.$

$\frac{df(x)}{dx} = (2x) . (\ln(3+2x)) + (x^2) . \left(\frac{1}{3+2x}\right) . (2)$

$= 2x\ln(3+2x) + \frac{2x^2}{3+2x}$

f) $y = \underbrace{x^3}\underbrace{e^x}\underbrace{\tan(x^2)}$ → product rule of 3 f^n

$\frac{dy}{dx} = \underbrace{(3x^2)}(e^x)(\tan(x^2)) + (x^3)\underbrace{(e^x)}(\tan(x^2))$

$+ (x^3)(e^x)\underbrace{(\sec^2(x^2) . 2x)}$

Using log diff

$\ln(y) = \ln(x^3 e^x \tan(x^2)) = 3\ln x + x + \ln(\tan(x^2))$

$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + 1 + \frac{\sec^2(x^2) . 2x}{\tan(x^2)}$

$\frac{dy}{dx} = \left[\frac{3}{x} + 1 + \frac{2x . \sec^2(x^2)}{\tan(x^2)}\right] . \underbrace{x^3 e^x \tan(x^2)}_{y}$

$= 3x^2 e^x \tan(x^2) + x^3 e^x \tan(x^2) +$

$2x . x^3 . e^x . \sec^2(x^2)$

g) $f(x) = e^{x^3 - \sin x}$

$f(x) = e^{g(x)}$

$g(x) = x^3 + \sin(x)$

$g'(x) = 3x^2 + \cos(x)$

$f'(x) = e^{g(x)} . g'(x)$

$f'(x) = e^{x^3 + \sin x} . (3x^2 + \cos(x))$

→ log diffentiation

h) $f(x) = \dfrac{(2-x)^2}{\sin x}$ → log diffentiation → quotient rule → product rule

$\dfrac{1}{\sin(x)} = \csc(x)$

$$f(x) = \underbrace{(2-x)^2}\cdot\underbrace{\csc(x)}$$

$$f'(x) = \underbrace{2(2-x)(-1)}\csc(x) + (2-x)^2\cdot(\underbrace{-\csc(x)\cdot\cot(x)})$$

i) $y = \log(\sin^2(5x))$ ⟶ Chain Rule 4 times

base = 10

$$\frac{dy}{dx} = \left[\frac{1}{\sin^2(5x)}\cdot\frac{1}{\ln(10)}\right]\left[2\sin(5x)\right]\left[\cos(5x)\right]\left[5\right]$$

j) $f(x) = \dfrac{7\pi}{\sqrt{5x+e^x}}$ $= (7\pi)(5x+e^x)^{-1/2}$

$$f'(x) = \underbrace{(7\pi)(-\tfrac{1}{2})}\underbrace{(5x+e^x)^{-3/2}}\cdot\underbrace{(5+e^x)}$$

Quotient rule

$$f'(x) = \frac{\cancel{\sqrt{5x+e^x}\left(\frac{d}{dx}(7\pi)\right)}^{\,0} - (7\pi)\frac{1}{2\sqrt{5x+e^x}}\cdot(5+e^x)}{(5x+e^x)}$$

$$= -\frac{7\pi}{2}\,\frac{1}{(5x+e^x)^{3/2}}(5+e^x)$$

Problem 2. If $f(x) = \sin^4(x)$, find $f'(\pi/3)$.

$$f'(x) = 4\sin^3(x)\cdot\cos(x)$$

$$f'\left(\frac{\pi}{3}\right) = 4\sin^3\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{3}\right)$$

$$f'\left(\frac{\pi}{3}\right) = 4\sin^3\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{3}\right) \leftarrow \text{Unit circle}$$

$$= 4\left(\frac{\sqrt{3}}{2}\right)^3 \cdot \frac{1}{2}$$

$$= \frac{\cancel{4}\cdot 3\sqrt{3}}{\underset{2}{\cancel{8}}\cdot 2} = \frac{3\sqrt{3}}{4} \text{ Ans.}$$

Problem 3. Find the 25^{th} derivative of $f(x) = xe^{-x}$.

$n = 25$

$$\frac{d}{dx}(e^{-x}) = -e^{-x}$$

$$f(x) = \underbrace{x}\,\underbrace{e^{-x}}$$

odd $n=1$ $\quad f'(x) = e^{-x} - \underbrace{xe^{-x}} = +1e^{-x} \underbrace{- xe^{-x}}$

even $n=2$ $\quad f''(x) = -e^{-x} - (e^{-x} - xe^{-x}) = -2e^{-x} + \underbrace{xe^{-x}}$

odd $n=3$ $\quad f^3(x) = +2e^{-x} + (e^{-x} - xe^{-x}) = +3e^{-x} \underbrace{- xe^{-x}}$

even $n=4$ $\quad f^4(x) = -3e^{-x} - (e^{-x} - xe^{-x}) = -4e^{-x} + xe^{-x}$

$$f^{25}(x) = +25e^{-x} - xe^{-x}$$

also, $f^{50}(x) = -50e^{-x} + xe^{-x}$

plug in t values

Problem 4. For the parametric curve given by $x = t^3 - 3t^2 - 9t + 1$, $y = t^3 + 3t^2 - 9t + 1$,

i) ii)

a) Find the point(s) (x, y) on the curve where the tangent lines are horizontal or vertical.

i) horizontal tangents $\Rightarrow \frac{dy}{dt} = 0$. $(m=0)$

$$y' = 3t^2 + 6t - 9 = 0$$
$$= 3(t^2 + 2t - 3) = 0$$
$$= \boxed{3(t+3)(t-1)} = 0$$

$(-26, 28) \leftarrow t = -3$ $\quad$ $t = 1 \rightarrow (-10, -4)$

ii) vertical tangent $\Rightarrow \frac{dx}{dt} = 0$ $\quad$ (m DNE)

$$x' = 3t^2 - 6t - 9 = 0$$
$$3(t^2 - 2t - 3) = 0$$
$$\boxed{3(t-3)(t+1)} = 0$$

$(-26, 28) \leftarrow t = 3$ $\quad$ $t = -1 \rightarrow (6, 12)$

b) Find the equation of the tangent line when $t = 2$.

$Pt\ (x, y)\big|_{t=2} \longrightarrow \boxed{(-21, 3)}$

$\Rightarrow$ slope m @ $t = 2$

$$m = \frac{y'}{x'}\bigg|_{t=2} = \frac{\cancel{3}(t+3)(t-1)}{\cancel{3}(t-3)(t+1)}\bigg|_{t=2}$$

$$= \frac{(5)(1)}{(-1)(3)} = \boxed{-\frac{5}{3}}$$

Eq^{n} of tangent line: $\boxed{y - 3 = \left(-\frac{5}{3}\right)(x + 21)}$

$m = \frac{dy}{dx}\Big|$ (or $^{-1}$)

$\rightarrow m = \frac{dy}{dx}\Big|_{(0,-1)}$

Problem 5. Find the slope of the tangent line to the curve $3y^3 - xy^2 - 3 = 0$ at the point $(0, -1)$.

why not $\frac{d}{dt}$? $\frac{dy}{dt}$ & $\frac{dx}{dt} \neq 1$

when you do $\frac{d}{dx}$ $\frac{dy}{dx}$ & $\frac{dx}{dx} = 1$

$$3y^3 - \underbrace{xy^2} + 3 = 0 \quad \leftarrow \text{Implicit differentiation}$$

$$3 \cdot 3y^2 \cdot \frac{dy}{dx} - \left[y^2 + x \cdot 2y\frac{dy}{dx} \right] + 0 = 0$$

$$9y^2 \cdot \frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0 .$$

$$\frac{dy}{dx}(9y^2 - 2xy) = y^2$$

$$\frac{dy}{dx} = \frac{y^2}{9y^2 - 2xy}\Bigg|_{\substack{x=0\\y=-1}} = \frac{(-1)^2}{9(-1)^2 - \cancel{2(0)}(-1)} = \frac{1}{9} = m$$

Ans.

Problem 6. For $f(x) = \frac{x^3+1}{x^2+1}$, find the equation of the tangent line at the point $x = -1$.

$$\text{pt } (x=-1) \longrightarrow (-1, f(-1)) = \left(-1, \frac{0}{2}\right) = (-1, 0)$$

Slope m $(x=-1)$

$$f'(x) = \frac{(x^2+1)(3x^2) - (x^3+1)(2x)}{(x^2+1)^2}\Bigg|_{x=-1}$$

$$= \frac{(2)(3) - \cancel{(0)}(\ldots)}{(1+1)^2} = \frac{6}{4} = \frac{3}{2}$$

Eqn:
$$y - 0 = \frac{3}{2}(x+1)$$

$$\boxed{y = \frac{3}{2}(x+1)}$$

$\vec{r}'(t=2)$ $|\vec{r}'(t=2)|$ $\vec{r}''(t=2)$

Problem 7. Find the velocity, speed and acceleration of each particle defined below: (t is given in seconds and distance is in meters.)

a) At time $t = 2$ when its position is given by $\vec{r}(t) =< \sqrt{t^2-5}, t >$. ← vector $\langle x, y \rangle$

$$\vec{r}'(t) = \left\langle \frac{\cancel{2}t}{\cancel{2}\sqrt{t^2+5}}, 1 \right\rangle$$

a) $\vec{r}'(2) = \left\langle \frac{2}{3}, 1 \right\rangle$

b) $|\vec{r}'(2)| = \sqrt{\left(\frac{2}{3}\right)^2 + 1}$

$$\vec{r}''(t) = \left\langle \frac{\sqrt{t^2+5}\,(1) - t\cdot\left(\frac{t}{\sqrt{t^2+5}}\right)}{t^2+5}, 0 \right\rangle$$

c) $\vec{r}''(2) = \left\langle \frac{3 - 4/3}{9}, 0 \right\rangle = \left\langle \frac{5}{27}, 0 \right\rangle$

b) At time $t = \pi/3$ when its position is given by $\vec{r}(t) =< 4\cos(2t), 3\sin(2t) >$.

$$\boxed{y(t) = y_0 e^{kt}}$$

exponential growth

Problem 8. The population of a bacteria culture grows at a rate proportional to its size. After 2 hours the bacteria population was 1000 and after 5 hours, the bacteria population was 7000. After how long will the bacteria culture attain a population of 35,000?

$t = ?$ $\quad$ $y(t)$

$$y(t=2) = 1000 = y_0 e^{2k} \quad \Big| \quad y(t=5) = 7000 = y_0 e^{5k}$$

$$y_0 = \frac{1000}{e^{2k}} = y_0 = \frac{7000}{e^{5k}}$$

$$\frac{1000}{e^{2k}} = \frac{7000}{e^{5k}} \Rightarrow \ln 7 = e^{5k-2k} \overset{\ln}{=} e^{3k} \to 3k \ln e = 3k$$

$$k = \frac{\ln 7}{3}$$

$$\boxed{y_0 = \frac{1000}{e^{2k}} = \frac{1000}{e^{\frac{2\ln 7}{3}}}}$$

$$e^{\ln 7^{2/3}} = 7^{2/3}$$

$$35\cancel{000} = \frac{1\cancel{000}}{e^{\frac{2}{3}\ln 7}} \cdot e^{\frac{t\ln 7}{3}} \quad \Big| \quad \ln\ 35 \cdot e^{\frac{2\ln 7}{3}} = e^{\frac{t}{3}\ln 7} = e^{\ln 7^{t/3}} \overset{\ln}{=} 7^{t/3}$$

$$\ln\left(35 \cdot e^{\frac{2\ln 7}{3}}\right) = \frac{t}{3}\ln 7$$

Problem 9. Find the half life of the radioactive isotope Strontium 90 if a sample decays to 95% of its original mass after 1 year.

formula. half life.

$$\boxed{t_{1/2} = \frac{\ln(1/2)}{k}} = \frac{\ln(0.5)}{k} = -\frac{\ln(2)}{k}.$$

Given $\boxed{t=1}$ yrs

$$\Rightarrow y(1) = \frac{95}{100} \cdot y_0 = 0.95 y_0$$

$$y(t) = y_0 e^{kt}$$

$$\ln\ 0.95 \cancel{y_0} = \cancel{y_0} e^{k}$$

$$\ln(0.95) = k$$

Ans.

$$t_{1/2} = \frac{\ln(0.5)}{k} = \boxed{\frac{\ln(0.5)}{\ln(0.95)}}$$

$$t = \frac{3\ln\left(35 \cdot e^{\frac{2\ln 7}{3}}\right)}{\ln 7} \text{ hours.}$$

$$= \frac{3\ln(35 \cdot 7^{2/3})}{\ln(7)}$$

$$= \frac{3\ln(35)}{\ln(7)} + \frac{\cancel{3} \cdot \frac{2}{\cancel{3}}\cancel{\ln(7)}}{\cancel{\ln(7)}}$$

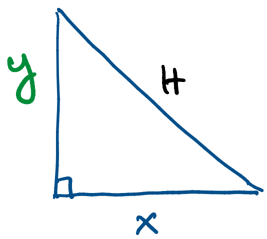
$$= \left[\frac{3\ln(35)}{\ln(7)} + 2\right] \text{hours.}$$

rate $\Rightarrow \frac{d}{dt}$

$\text{rate} \Rightarrow \frac{d}{dt}$

Problem 10. Consider a right angled triangle. If the horizontal leg of the trangle is increasing at the rate of 5 m/s and the vertical leg of the triangle is decreasing at the rate of 6 m/s, at what rate is the hypotenuse changing when the horizontal leg is 12m and the vertical leg is 9m ?

Snapshot.



$\frac{dy}{dt} = -6\,m/s$

$\frac{dx}{dt} = 5\,m/s$

Q: $\left.\frac{dH}{dt}\right|_{\substack{x=12\\y=9}} = ?$

$$\frac{d}{dt}\left[H^2 = x^2 + y^2\right]$$

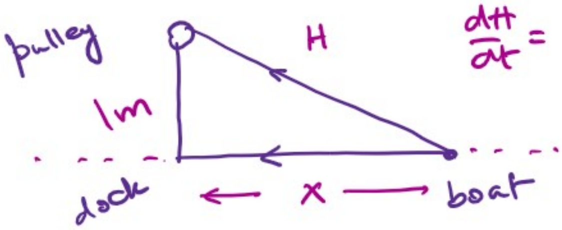
$$\cancel{2}H\frac{dH}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\left.\frac{dH}{dt}\right|_{\substack{x=12\\y=9}} = \frac{1}{H}?\left[\cancel{x}^{12}\,\cancel{\frac{dx}{dt}}^{5} + \cancel{y}^{9}\,\cancel{\frac{dy}{dt}}^{-6}\right].$$

$$H = \sqrt{x^2+y^2} = \sqrt{(12)^2+(9)^2} = \sqrt{144+81} = \sqrt{225} = 15$$

$$\frac{dH}{dt} = \frac{1}{15}\left[60 - 54\right] = \frac{6}{15} = \frac{2}{5}\,m/s$$

Problem 11. A boat is being pulled into a dock by a pulley that is fixed 1m above the water level, at a rate of 1 m/s. How fast is the boat approaching the dock when the boat is 8 meters away from the dock?



$$\frac{dH}{dt} = -1\,m/s$$

Q: $\left.\frac{dx}{dt}\right|_{x=8} = ?$

$$\frac{d}{dt}\left[H^2 = x^2 + 1\right]$$

$$\cancel{2}H\frac{dH}{dt} = \cancel{2}x\frac{dx}{dt} + 0$$

$$\left.\frac{dx}{dt}\right|_{x=8} = \frac{H}{x}\frac{dH}{dt}$$

? (-1)

8

$$H = \sqrt{x^2+1} = \sqrt{64+1} = \sqrt{65}$$

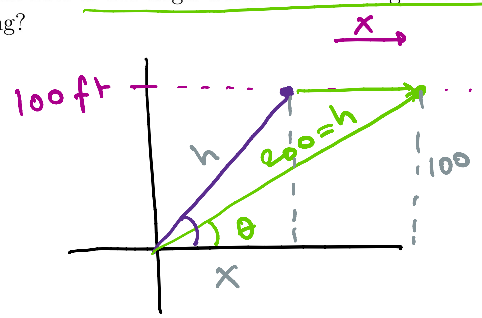
$$\left.\frac{dx}{dt}\right|_{x=8} = \frac{\sqrt{65}}{8}(-1)$$

$$= -\frac{\sqrt{65}}{8}\ m/s$$

distance is shrinking between boat & dock.

Problem 12. A kite flying 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizon decreasing when the string is 200 ft long?

snapshot



$$\frac{dx}{dt} = 8 \text{ ft/s}$$

$$Q: \left.\frac{d\theta}{dt}\right|_{h=200} = ?$$

★ $$\frac{d}{dt}\left[\sin\theta = \frac{100}{h}\right]$$

$$\underbrace{\cos\theta\,\frac{d\theta}{dt} = (100)\left(-\frac{1}{h^2}\,\frac{dh}{dt}\right)}_{\text{not the best}}$$

??

★ $$\cos\theta = \frac{x}{h}$$

$$\underbrace{\frac{dx}{dt} \qquad \frac{dh}{dt}}_{\text{not this either}}$$

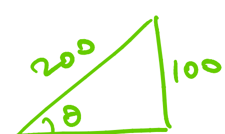
??

★ $$\frac{d}{dt}\left[\tan\theta = \frac{100}{x}\right]$$

$$\sec^2\theta\,\frac{d\theta}{dt} = 100\left(-\frac{1}{x^2}\,\frac{dx}{dt}\right)$$

$$\left.\frac{d\theta}{dt}\right|_{h=200} = \frac{-100}{\sec^2\theta \cdot x^2} \cdot \cancel{\frac{dx}{dt}}\ 8$$

$$\frac{1}{\sec\theta} = \cos\theta = \frac{\sqrt{30{,}000}}{200}$$



$$x = ? = \sqrt{(200)^2 - (100)^2}$$
$$= \sqrt{40000 - 10000}$$
$$x = \sqrt{30{,}000}$$

$$\left.\frac{d\theta}{dt}\right|_{h=200} = \frac{(-100)}{(\cancel{30000})} \cdot \frac{(\cancel{30000})}{(40{,}000)} \cdot 8 = \boxed{-\frac{1}{50} \text{ rad/s}}$$ Ans.

$$L(x) = f(a) + f'(a)(x-a)$$

a = value you know

x = value you will estimate

$a =$ value you know

$x =$ value you will estimate

a) b)

Problem 13. Find the linear approximation for $f(x) = \dfrac{1}{\sqrt{4+x}}$ at $a = 0$. Use the linear approximation model to estimate the value of $\dfrac{1}{\sqrt{4.01}}$.

$$f(a) = f(0) = \frac{1}{\sqrt{4+0}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$f'(x) = \frac{d}{dx}(4+x)^{-1/2} = -\frac{1}{2}(4+x)^{-3/2} \cdot (1)$$

$$f'(a) = f'(0) = -\frac{1}{2}(4)^{-3/2} = \left(-\frac{1}{2}\right)\left(\frac{1}{8}\right) = -\frac{1}{16}$$

$$(2^2)^{-\frac{3}{2}} = 2^{-3} = \frac{1}{8}$$

a) **Model:** $L(x) = \frac{1}{2} - \frac{1}{16}x$

b) $\underbrace{x = ?}_{x = 0.01}$

$$f(x) = \frac{1}{\sqrt{4+x}} = \frac{1}{\sqrt{4.01}}$$

$$4 + x = 4.01$$

$$x = 0.01$$

$$L(x) = L(0.01) = \frac{1}{2} - \frac{1}{16}(0.01)$$

$$\therefore \frac{1}{\sqrt{4.01}} = \frac{1}{2} - \frac{0.01}{16}$$

$$\sim 0.499$$

$$L(x) = f(a) + f'(a)(x-a)$$

$x = 8.012$

Problem 14. Use linear approximation to estimate the value of $\sqrt[3]{8.012}$.

$f(x) = \sqrt[3]{x} = x^{1/3}$

a good value for $a = 8$

$\therefore f(8) = \sqrt[3]{8} = 2$

$f'(x) = \frac{1}{3} x^{-2/3}$

$f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}(2^3)^{-2/3} = \frac{1}{3}(2^{-2}) = \frac{1}{12}$

$L(x) = L(8.012) = 2 + \frac{1}{12}(8.012 - 8) = 2 + \frac{0.012}{12}$

$= 2.001$ Ans.

Problem 15. The radius of a sphere was measured to be 5 cm with a maximum error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in the calculated volume of the sphere. What is the percentage relative error?

a) dV

$\boxed{r = 5 \text{ cm}}$ $\boxed{dr = 0.1 \text{ cm}}$

$V = \frac{4}{3}\pi r^3 \rightarrow V = \frac{4}{3}\pi(125) = \frac{500\pi}{3}$

$\frac{dV}{dr} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2 = 4\pi r^2$

$dV = (4\pi r^2)dr$

$= (4\pi\, 25)(0.1)$

$= (100\pi)(0.1)$

$dV = 10\pi \rightarrow$ max error in volume.

$\boxed{V = \frac{500\pi}{3} \pm 10\pi}$

real world example.

b) % rel. error $= \frac{dV}{V} \cdot 100 = \left(\frac{10\cancel{\pi}}{\frac{500\cancel{\pi}}{3}}\right)(100) = \frac{30}{5} = 6\%$