WIR 8

TEXAS A&M UNIVERSITY
Math Learning Center

Math 151 - Fall 2022
"HANDS ON, GRADES UP"

WEEK-IN-REVIEW 8: CHAPTER K2, 3.7, 3.8
(DERIVATIVES OF PARAMETRIC EQUATIONS AND APPLICATIONS OF DERIVATIVES.)

Problem 1. Find the equation(s) of the tangent line(s) to the parametric curve given by $x = 3t^2 + 1,\ y = 2t^3 + 1,$ that pass through the point $(4, 3)$.

$x = 3t^2 + 1$ $\qquad$ $y = 2t^3 + 1$

$\frac{dx}{dt} = 6t$ $\qquad$ $\frac{dy}{dt} = 6t^2$

$m = \frac{dy/dt}{dx/dt} = \frac{y'}{x'} = \frac{6t^2}{6t} = t$ $\qquad$ $\boxed{\therefore m = t = 1}$

$\boxed{\text{Pt } (4,3)}$ $\longrightarrow$ what is t here?

$x = 4$ $\qquad$ $x = 3t^2 + 1$

$3t^2 + 1 = 4$

$3t^2 = 3$

$t^2 = 1$

$t = +1$ or -1

$y = 3$ $\qquad$ $y = 2t^3 + 1$

$2t^3 + 1 = 3$

$2t^3 = 2$

$t^3 = 1$

$t = 1$

Eqn: $y - 3 = 1(x - 4)$

or, $y = x - 1$

Problem 2. Find the slope of the tangent line to the curve given by $x = \sin(3s^2)$, $y = s\ln(1-s)$.

$$m = \frac{dy}{dx} = \frac{dy/ds}{dx/ds}$$

$\frac{d}{ds}(1-s) = -1$

$$x = \sin(3s^2)$$

$$\frac{dx}{ds} = \cos(3s^2)\cdot(6s)$$

$$y = s\ln(1-s)$$

$$\frac{dy}{ds} = \ln(1-s) + s\cdot\frac{(-1)}{1-s}$$

$$m = \frac{\ln(1-s) - \frac{s}{1-s}}{6s\cos(3s^2)} = \frac{(1-s)\ln(1-s) - s}{6s(1-s)\cdot\cos(3s^2)}$$

Problem 3. Find the equation of the tangent line to the curve $x = 5e^{t^2}$, $y = t^2+3$, when $t = 2$.

$$m\Big|_{t=2} = \frac{dy/dt}{dx/dt}\Big|_{t=2} = \frac{2t}{5(e^{t^2})(2t)}\Big|_{t=2}$$

$$m\Big|_{t=2} = \frac{\cancel{4}}{5(e^4)(\cancel{4})} = \frac{1}{5e^4}$$

Point $\rightarrow$ $t=2$ $\rightarrow$

$x = 5e^{t^2} = 5e^4$

$y = t^2+3 = 4+3 = 7$

$(5e^4, 7)$

Eqn:
$$y - 7 = \left(\frac{1}{5e^4}\right)(x - 5e^4)$$

$$y - 7 = \frac{x}{5e^4} - 1$$

$$y = \frac{x}{5e^4} + 6$$

$$m = \frac{dy/dt}{dx/dt}$$

Problem 4. Find the point(s) on the curve where the tangent lines are horizontal or vertical.

a) $x = t^3 - 3t^2$, $y = t^3 - 3t$

horizontal $\rightarrow$ $m=0$

vertical $\rightarrow$ m DNE

$\boxed{\text{Horizontal:}}$ $\frac{dy}{dt} = 0$.

a) $x = t^3 - 3t^2$, $y = t^3 - 3t$

$m = 0 \downarrow \frac{dy}{dt} = 0$ $\qquad$ m DNE $\downarrow \frac{dx}{dt} = 0$.

$\boxed{\text{Horizontal:}}$ $\frac{dy}{dt} = 0$.

$3t^2 - 3 = 0$.

$3(t^2-1) = 0$.

$3(t+1)(t-1) = 0$.

$t=-1$ $\quad$ $t=+1$

$t=-1$

$(-1-3, -1+3) = \boxed{(-4, 2)}$

$t = 1 \to (1-3, 1-3) = \boxed{(-2,-2)}$

$\boxed{\text{Vertical}}$ $\frac{dx}{dt} = 0 \Rightarrow 3t^2 - 6t = 0$

$3t(t-2) = 0$

$t=0$, $t=2$

$t=0 \Rightarrow \boxed{(0,0)}$

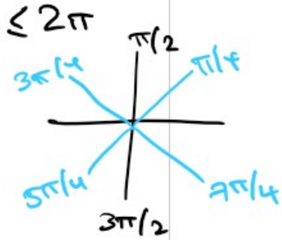
$t=2 \Rightarrow (8-12, 8-6) = \boxed{(-4,2)}$

b) $x = \sin(2t)$, $y = \sin(t)$

Horizontal : $\frac{dy}{dt} = 0$.

$0 \le t \le 2\pi$

$\cos(t) = 0$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$



$t = \pi/2 \to (0,1)$

$t = 3\pi/2 \to (0,-1)$

Vertical : $\frac{dx}{dt} = 0$

$2\cos(2t) = 0$

$\cos(2t) = 0$ when $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$0 \le t \le 2\pi$

$0 \le 2t \le 4\pi$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$t = \frac{\pi}{4} \to (1, \frac{\sqrt{2}}{2})$

$t = \frac{3\pi}{4} \to (-1, \frac{\sqrt{2}}{2})$

$t = \frac{5\pi}{4} \to (1, -\frac{\sqrt{2}}{2})$

$t = \frac{7\pi}{4} \to (-1, -\frac{\sqrt{2}}{2})$

Problem 5. The path of a moving particle is defined by its position function, $r(t) = 2t^3 - 12t^2 + 18t - 3$, $t > 0$. (t is in seconds and r is in meters.)

a) When is the particle at rest?

$\underbrace{v(t)} = r'(t) = 0$.

$r'(t) = \boxed{6t^2 - 24t + 18} = 0$.

$t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$.

$t=3$ $\quad$ $t=1$

At rest when $t=1$ and $t=3$ seconds

b) When is the particle moving forward? When is the particle moving backwards?

$v > 0$ $\qquad$ $v < 0$

$v(\frac{1}{2}) = 6(\frac{1}{4}) - 24(\frac{1}{2}) + 18 > 0$

$v(2) = 6(4) - 24(2) + 18 < 0$

$v(5) = 6(25) - 24(5) + 18 > 0$

$t=0$ $\quad \frac{1}{2} \quad$ $t=1$ $\quad 2 \quad$ $t=3$ $\quad 5$

v: $+$ $\quad$ $-$ $\quad$ $+$

$v(2) = 6(4) - 24(2) + 18 < 0$

$v(5) = 6(25) - 24(5) + 18 > 0$

$t=0 \quad \tfrac{1}{2} \quad t=1 \quad 2 \quad t=3 \quad 5$

$v: \quad + \quad - \quad +$

c) Find the total distance travelled by the particle in the first 5 seconds.

forward: $[0,1) \cup (3,\infty)$

backward: $(1,3)$

1\. $|r(1) - r(0)| = (2-12+18+3) - (3) = 8m$

2\. $|r(3) - r(1)| = (54 - 108 + 54 + 3) - (11)$

$= |-8| = 8m$

3\. $|r(5) - r(3)| = (250 - 300 + 90 + 3) - (3)$

$= 40m$

Total distance travelled $= 8 + 8 + 40 = 56m$

d) What is the diplacement of the particle after 5 seconds?

Displacement: $r(5) - r(0) = 43 - 3$

$= 40m$

Exponential growth / decay

$$y(t) = y_0 \cdot e^{kt}$$

$k \to$ const $\to$ + for growth

$\to$ − for decay

Problem 6. A laboratory experiment finds that the population of a bacteria culture grows at a rate proportional to its size. The bacteria population is found to triple every 5 hours.

$\to$ solve for k.

a) Find a function that models how much bacteria there will be after t hours.

$t = 5$ $\quad y(5) = 3y_0$

$$3\cancel{y_0} = \cancel{y_0} \cdot e^{5k}$$

$$\ln(3) = \ln e^{5k} = 5k \Rightarrow k = \frac{\ln(3)}{5}$$

$$y(t) = y_0 e^{t\frac{\ln 3}{5}} = y_0 e^{\ln 3^{t/5}} = y_0 \cdot 3^{t/5}$$

$$y(t) = (1000) \cdot 3^{t/5}$$

$e^{(\frac{t}{5})\ln 3}$

$\boxed{a \ln x = \ln x^a}$

$e^{\ln 3^{t/5}}$

$\boxed{e^{\ln x} = x}$

$3^{t/5}$

b) How much bacteria will the sample contain after 10 hours?

$t = 10$

$$y(10) = 1000 \cdot 3^{10/5} = 1000\,(3^2)$$

$$= 9000$$

c) If the culture started with a population of 1000, after how long will the culture attain a population of 30,000?

$y(t)$ $\quad \to$ solve for t

$$30{,}\cancel{000} = (1\cancel{000}) \cdot 3^{t/5}$$

$$\ln(30) = \ln 3^{t/5} = \frac{t}{5} \ln 3$$

$$t = 5\,\frac{\ln(30)}{\ln(3)} \text{ hours. } (\sim 15.5 \text{ hrs})$$

$$\left.\begin{array}{l} y(t) = y_0 e^{kt} \\ \text{half life}: \quad y(t) = \frac{1}{2} y_0 \\ t = t_{1/2} \end{array}\right\}$$

$$\frac{1}{2}\cancel{y_0} = \cancel{y_0} e^{kt_{1/2}}$$

$$\ln(\tfrac{1}{2}) = \ln e^{kt_{1/2}} = kt_{1/2}$$

$$\boxed{k = \frac{\ln(1/2)}{t_{1/2}}} = \frac{-\ln 2}{t_{1/2}}$$

$\Downarrow$

Problem 7. Strontium 90 is a radioactive isotope with a half life of 25 years.

a) If there are 20 mgs of the isotope today, how much of it will be left after 15 years?

y_0 $\quad y(t) = 20e^{15k}$ $\quad \to y(t) = ?$

a) If there are 20 mgs of the isotope today, how much of it will be left after 15 years?

y_0 $\quad y(t) = 20e^{15k}$ $\quad y(t) = ?$

$$k = -\frac{\ln 2}{t_{1/2}} = -\frac{\ln 2}{25}$$

$$y(15) = (20)\cdot e^{\cancel{15}^{3}\cdot\left(-\frac{\ln 2}{\cancel{25}_{5}}\right)} = 20e^{-\frac{3}{5}\ln 2}$$

$$= 20\left(2^{-3/5}\right)$$

$$= 13.195 \text{ mg}$$

b) After how many years will there be 2 mgs of the substance left?

solve for t $\quad y(t)$

$$\underline{2} = (20)\cdot e^{kt} = (20)e^{t\left(-\frac{\ln 2}{25}\right)}$$

$$= (20)e^{-\frac{t}{25}\ln 2}$$

$$\frac{1}{10} = \frac{2}{20} = \cancel{(20)}\left(2^{-t/25}\right)$$

$$\underbrace{\ln\left(\frac{1}{10}\right)}_{-\ln 10} = \ln\left(2^{-t/25}\right) = -\frac{t}{25}\ln 2.$$

$\cancel{\ln 1}^{0}$

$$\ln(10) = \frac{t}{25}\ln(2)$$

$$t = 25\cdot\frac{\ln(10)}{\ln(2)} \text{ yrs} = 83.05 \text{ years..}$$

Problem 8. Iodine-131 has a half life of 8 days. How much of a 40 gram sample wil be left after 48 hours?

$y(t=2) = ?$ $\quad y_0$

$t = 2$

$$t_{1/2} = 8 \text{ days} \qquad \therefore k = -\frac{\ln 2}{8}$$

$$y(2) = (40)\,e^{2\cdot\left(-\frac{\ln 2}{8}\right)} = 40e^{-\frac{1}{4}\ln 2}$$

$$= (40)\left(2^{-1/4}\right) \text{ grams.}$$

$$= (40)(0.84)$$

$$= 33.6 \text{ grams}$$

$$= (40)(0.84)$$

$$= 33.6 \text{ grams.}$$

Problem 9. If you have 100 grams of a radioactive isotope with a half-life of 10 years, how much of the isotope will you have left after 10 years? After 20 years? After 40 years?

$$t_{1/2} = 10 \text{ years}$$

$$y_0 = 100\ g$$

$$y(10) = \frac{1}{2}(100) = 50\ g$$

$$y(20) = \frac{1}{2}(50) = 25g$$

$$y(30) = \frac{1}{2}(25) = 12.5g$$

$$y(40) = \frac{1}{2}(12.5) = 6.25g$$

T: temperature of reservoir (surrounding) → constant

$$y(t) - T = (y_0 - T)e^{kt}$$

Problem 10. Newton's Law of Cooling states the rate of cooling of an object is proportional to the temperature difference between the object and the temperature of the object's surroundings. A pie is taken from an oven where the temperature is 375° F and placed on a table in a room where the temperature is 75° F. After 20 minutes, the temperature of the pie is 200° F.

decay.

a) Find a formula for the temperature of the pie at time t, where t is measured in minutes.

$$\left.\begin{array}{l} T = 75 \\ y_0 = 375 \\ y(t) = 200 \end{array}\right\} \text{solve for } k \text{ first}$$

$$(200 - 75) = (375 - 75)e^{20k}$$

$$125 = 300e^{20k} \Rightarrow \frac{125}{300} = \frac{5}{12} = e^{20k}$$

$$k = \frac{1}{20}\ln\left(\frac{5}{12}\right)$$

formula: $y(t) - 75 = 300e^{\frac{t}{20}\ln\left(\frac{5}{12}\right)}$

b) After how long will the pie reach a temperature of 100° F?

solve for t

→ $y(t)$

$$100 - 75 = 300e^{\left(\frac{t}{20}\right)\ln\left(\frac{5}{12}\right)}$$

$$25 = 300\left(\frac{5}{12}\right)^{t/20}$$

$$\left(\frac{1}{12}\right) = \left(\frac{5}{12}\right)^{t/20}$$

$$\ln\left(\frac{1}{12}\right) = \frac{t}{20}\ln\left(\frac{5}{12}\right)$$

$$t = 20\frac{\ln(1/12)}{\ln(5/12)} = 56.77 \text{ minutes}$$

Problem 11. A spherical balloon is being inflated. Find the rate at which the volume of the balloon is changing with respect to the radius when the radius of the balloon is 2 inches and when the radius is 4 inches.

volume of a sphere: $\frac{4}{3}\pi r^3 = V$.

balloon is changing with respect to the radius when the radius of the balloon is 2 inches and when the radius is 4 inches.

volume of a sphere: $\frac{4}{3}\pi r^3 = V$.

r = radius

$$\frac{dV}{dr} = \left(\frac{4}{3}\pi\right)\frac{d}{dr}\cdot r^3 = \frac{4}{3}\pi\left(3r^2\right)$$

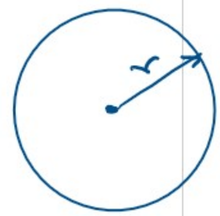
$$\left.\frac{dV}{dr}\right|_{r=2} = 4\pi r^2 = 4\pi(4) = 16\pi$$

$$\left.\frac{dV}{dr}\right|_{r=4} = 4\pi r^2 = 4\pi(16) = 64\pi$$

Problem 12. A stone dropped in a pond sends out a circular ripple whose radius increases at a constant rate of 4 ft/sec. How rapidly is the area enclosed by the ripple increasing after 12 seconds?

radius = r

$$4 \text{ ft/s} = \frac{dr}{dt} = \text{const}$$



$$\boxed{A = \pi r^2}$$

Implicit diff

$$\frac{dA}{dt} = \pi\frac{d}{dt}\left(r^2\right)$$

$\frac{dr}{dt} = 4$

@ $t = 12$

$r = 4 * 12 = 48$ ft

$$\left.\frac{dA}{dt}\right|_{t=12} = \pi\,(2r)\cdot\left(\frac{dr}{dt}\right)$$

$$= \pi\cdot 2\,(48)\,(4)$$

$$\left.\frac{dA}{dt}\right|_{t=4} = 384\pi \text{ ft}^2/\text{second}.$$