WIR 7

Week-in-Review 7: Chapter 3.5, 3.6, K1
(Implicit Diffentiation, Derivatives on Inverse Trigonometric, Logarithmic and Vector Functions.)

Problem 1. Find the following derivatives:

$y \neq f(x) \rightarrow$ implicit-differenciation

(a) $\frac{d}{dx}\left[9y^4 - 12x^2y^2 + 5x^2 - 11x\right]$

$y' = \frac{dy}{dx}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(y^n) = ny^{n-1} \cdot \frac{dy}{dx}$

$\frac{d}{dx}(y) = \frac{dy}{dx} = y'$

① $9(4y^3)\cdot\frac{dy}{dx} - 12\left[x^2\cdot(2y)\frac{dy}{dx} + y^2\cdot(2x)\right] + 5(2x) = 11$

② $\frac{dy}{dx}\left[36y^3 - 24x^2y\right] = 11 + 24xy^2 - 10x$

③ Ans: $y' = \frac{dy}{dx} = \frac{11 + 24xy^2 - 10x}{36y^3 - 24x^2y}$

(b) $\sqrt{y}\cos x + \sin(3y) - \cot^2(3x) = 1$

Answer contains terms with x & y in them

$\frac{d}{dx}\left[\sqrt{y}\cos(x) + \sin(3y) - \cot^2(3x)\right] = \frac{d}{dx}(1)$

① $\sqrt{y}(-\sin(x)) + \frac{\cos(x)}{2\sqrt{y}}\left(\frac{dy}{dx}\right) + 3\cos(3y)\frac{dy}{dx} - 2\cot(3x)\cdot(-\csc^2(3x))\cdot(3) = 0$

$\frac{dy}{dx}\left[\frac{\cos(x)}{2\sqrt{y}} + 3\cos(3y)\right] = \sqrt{y}\sin(x) - 6\cot(3x)\cdot\csc^2(3x)$

Ans: $\boxed{y' = \frac{dy}{dx} = \frac{\sqrt{y}\sin(x) - 6\cot(3x)\csc^2(3x)}{\frac{\cos(x)}{2\sqrt{y}} + 3\cos(3y)}}$

$$\left(\frac{x^2+y^2-2y}{x^2+y^2}\right)$$

$\ln(AB) = \ln(A) + \ln(B)$

but $\ln(A+B)$ cant be split further

$y = f(x)$

(c) $f(x) = \ln(x^2 + y^2)$ $\Rightarrow$ $\frac{d}{dx}\left[y = \ln(x^2+y^2)\right]$ $\Rightarrow$ Implicit diff.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{(x^2+y^2)}\frac{d}{dx}(x^2+y^2) = \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[1 - \frac{2y}{x^2+y^2}\right] = \frac{2x}{x^2+y^2}$$

Ans.

$$\frac{dy}{dx} = \frac{\left(\frac{2x}{x^2+y^2}\right)}{\left(\frac{x^2+y^2-2y}{x^2+y^2}\right)} = \frac{2x}{x^2+y^2-2y} = y'$$

$b = e$ (d) $f(x) = \ln(x^2 e^{-2x})$

$$f'(x) = \frac{1}{(x^2 \cdot e^{-2x})} \cdot \frac{d}{dx}\left(x^2 \cdot e^{-2x}\right)$$

$$= \frac{(2x)(e^{-2x}) + (x^2)(e^{-2x})\cdot\frac{d}{dx}(-2x)}{x^2\cdot e^{-2x}}$$

with $\frac{d}{dx}(-2x) = (-2)$

$$\boxed{f'(x) = \frac{2x\cdot e^{-2x} - 2x^2 e^{-2x}}{x^2 \cdot e^{-2x}}}$$ Ans.

$$\frac{d}{dx}(e^x) = e^x \cdot \frac{d}{dx}(x) = e^x$$

$$\frac{d}{dx}(e^{2x}) = e^{2x}\cdot\frac{d}{dx}(2x) = 2e^{2x}$$

$$\frac{d}{dx}(e^{-2x}) = e^{-2x}(-2)$$

chain Rule

$\boxed{b=10}$ (e) $f(x) = \tan[\log(ax + b)]$

a, b = constants

$$f'(x) = \sec^2[\log(ax+b)] \cdot \frac{d}{dx}[\log(ax+b)]$$

$$= \sec^2[\log(ax+b)] \cdot \frac{1}{\ln(10)(ax+b)} \cdot \frac{d}{dx}(ax+b)$$

$$= \underbrace{\left(\sec^2[\log(ax+b)\right)}\underbrace{\left(\frac{1}{(ax+b)\ln(10)}\right)}\underbrace{(a)}$$

$\rightarrow$ Chain Rule.

$$\log_a(x^n) = n\log_a(x)$$

$b=2$ (f) $f(x) = \log_2(x^3+6x)^5 = 5\log_2(x^3+6x)$

$$y' = f'(x) = 5 \cdot \frac{1}{(x^3+6x)\cdot\ln(2)}\cdot(3x^2+6)$$

$$\boxed{f'(x) = \frac{5(3x^2+6)}{(x^3+6x)\cdot\ln(2)}}$$ Ans.

(g) $f(x) = x\ln(\sin(3x))$

$b=e$

$$f'(x) = \frac{d}{dx}(x)\cdot\ln(\sin(3x)) + x\cdot\frac{d}{dx}\ln(\sin(3x))$$

$$= \ln(\sin(3x)) + x \cdot \frac{1}{\sin(3x)} \cdot \cos(3x) \cdot 3$$

(h) $f(x) = \arcsin(e^x)$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{1}{\sqrt{1-(e^x)^2}}\cdot\frac{d}{dx}(e^x)$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

$$f'(x) = \frac{e^x}{\sqrt{1-e^{2x}}}$$

$$(e^x)^2 = e^x \cdot e^x = e^{x+x} = e^{2x}$$

(i) $f(x) = \tan^{-1}(5x^2)$

$$f'(x) = \frac{1}{1+(5x^2)^2}\cdot\frac{d}{dx}(5x^2) \rightarrow 5(2x)$$

$$(5x^2)^2 = 5x^2 \cdot 5x^2 = 25 \cdot x^4$$

$$= \frac{10x}{1+25x^4}$$

$$\ln(AB) = \ln A + \ln B = \ln(AB)$$

$$* \ \ln\left(\frac{A}{B}\right) = +\ln A - \ln B = \ln\left(\frac{A}{B}\right)$$

$$\ln(x^n) = n\ln(x)$$

$y=f(x)$

$b=e$ (j) $f(x) = \ln\left(\frac{x^2+1}{5xe^x(x^3+11)^4}\right) = \ln\left(\frac{(x^2+1)}{(5)(x)(e^x)(x^3+11)^4}\right)$

$$\ln(x^2+1) - \ln(5) - \ln(x) - \ln(e^x) - \ln(x^3+11)^4$$

$b=e$

(j) $f(x) = \ln\left(\frac{x^2+1}{5xe^x(x^3+11)^4}\right)$

$$y = \ln(x^2+1) - \ln(5) - \ln(x) - \ln(e^x) - \ln(x^3+11)^4$$

$$= \ln(x^2+1) - \underbrace{\ln(5)}_{c} - \ln(x) - \underbrace{x(\ln e)}_{x} - 4\ln(x^3+11)$$

$$y' = \frac{1}{x^2+1}\cdot(2x) - 0 - \frac{1}{x} - 1 - 4\cdot\frac{1}{x^3+11}\cdot(3x^2)$$

$$\boxed{y' = \frac{2x}{x^2+1} - \frac{1}{x} - 1 - \frac{12x^2}{x^3+11}}$$ Ans.

(k) $f(x) = \left\langle \underbrace{\frac{\frac{1}{x}-\frac{1}{3}}{x-3}}_{x}, \underbrace{2x-3}_{y} \right\rangle$. State the domain of f and f'.

all x ok.

$$f'(x) = \left\langle \frac{d}{dx}\left(-\frac{1}{3x}\right), \frac{d}{dx}(2x-3) \right\rangle$$

$$f'(x) = \left\langle \frac{1}{3x^2}, 2 \right\rangle$$

Domain of $f'(x)$: $x \neq 0$

$$\frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \frac{\left(\frac{3-x}{3x}\right)}{(x-3)} = \frac{\cancel{(3-x)}}{3x}\cdot\frac{1}{\cancel{(x-3)}} = \frac{-1}{3x}$$

(-1)

Dom of $f(x)$: $x \neq 3$, $x \neq 0$

$$\frac{d}{dx}\left(-\frac{1}{3x}\right) = -\frac{1}{3}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{+1}{3x^2}$$

$(x)^x$ $\nearrow \neq$ power f^{n} (x in base)
$\searrow \neq$ exponential f^{n} (x in exp)

Problem 2. Use log differentiation to find the following derivatives.

a $f(x) = x^x$

1. $\boxed{y = x^x}$

2. $\ln(y) = \ln(x^x)$
$= \underbrace{x}\,\underbrace{\ln(x)}$

3. $\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \left(\frac{1}{x}\right) + (1) \cdot \ln(x) = 1 + \ln(x)$

4. $\frac{dy}{dx} = (1 + \ln(x)) \cdot \boxed{y}$

5. $\frac{dy}{dx} = [1 + \ln(x)] \cdot (x^x)$

b $f(x) = (\sin x)^{\cos x}$

1. $y = (\sin x)^{\cos x}$

2. $\ln(y) = \ln\left(\sin x\right)^{\cos(x)}$
$= \underbrace{\cos(x)} \cdot \underbrace{\ln(\sin x)}$

3. $\frac{1}{y} \cdot \frac{dy}{dx} = \cos(x) \cdot \frac{1}{\sin(x)} \cdot \cos(x) + (-\sin x) \cdot \ln(\sin(x))$

4 & 5. $\frac{dy}{dx} = \left[\frac{\cos^2(x)}{\sin(x)} - \sin(x) \ln(\sin(x))\right] \cdot \left(\underbrace{\sin(x)^{\cos(x)}}_{y}\right)$

Problem 3. Find the slope of the tangent line to the curve $\sec(x + y) - \tan(x - y) = 1$ at the point (π, π).

$$\frac{d}{dx}\left[\ \sec(x+y) - \tan(x-y) = 1\ \right] \leftarrow \text{implicit}$$

$$\sec(x+y)\cdot\tan(x+y)\underbrace{\left[\cdot\frac{d}{dx}(x+y)\right]}_{1+\frac{dy}{dx}} - \sec^2(x-y)\underbrace{\left[\cdot\frac{d}{dx}(x-y)\right]}_{1-\frac{dy}{dx}} = 0$$

$$\sec(x+y)\cdot\tan(x+y)\frac{dy}{dx} + \sec^2(x-y)\frac{dy}{dx} + \sec(x+y)\tan(x+y) - \sec^2(x-y) = 0.$$

$$\frac{dy}{dx}\left[\sec(x+y)\cdot\tan(x+y) + \sec^2(x-y)\right] = \sec^2(x-y) - \sec(x+y)\tan(x+y)$$

$$m = \frac{dy}{dx} = \frac{\overset{1}{\cancel{\sec^2(x-y)}} - \overset{1}{\cancel{\sec(x+y)}}\,\overset{0}{\cancel{\tan(x+y)}}}{\underset{1}{\cancel{\sec(x+y)}}\,\underset{0}{\cancel{\tan(x+y)}} + \underset{1}{\cancel{\sec^2(x-y)}}}\Bigg|_{\substack{x=\pi\\ y=\pi}}$$

$$\sec^2(x-y) = \sec^2(0) = \frac{1}{\cos^2(0)} = 1$$

$$\sec(x+y) = \frac{1}{\cos(2\pi)} = 1$$

$$\tan(x+y) = \tan(2\pi) = 0$$

$$m\Big|_{(\pi,\pi)} = \frac{1-0}{0+1} = 1.$$

$$\boxed{\text{Ans: } m = 1}$$

Problem 4. Find the equation of the tangent line to the curve $y = x^2 \ln(x)$ at the point $(1, 0)$.

$b = e$

$$y = x^2 \ln(x) \leftarrow \text{NOT implicit}$$

$b = e$

$y = \underbrace{x^2}\underbrace{\ln(x)}$ ← NOT implicit

Point: $(1,0)$

Slope $m = 1.$

$y' = (x^2)\cdot\frac{1}{x} + (2x)\cdot\ln(x)$

$m = y' = x + 2x\ln(x)\Big|_{x=1}$

$\log_a 1 = 0$

$m = 1 + \underbrace{2(1)\cdot\cancel{\ln}(1)}_{=0}$

$m = 1$

Eqn : $y - 0 = 1(x-1)$

$\boxed{y = x - 1}$ Ans

Problem 5. Find a tangent vector of length 5 to the curve $\vec{r}(t) = (2t\sin t)\vec{i} - (3 - 4\cos(t))\vec{j}$ at the point where $t = \frac{\pi}{2}$

$$\frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\vec{r}(t) = \langle 2t\sin t,\ 3 - 4\cos t \rangle.$$

$$r'(t) = \langle (2\sin t + 2t\cos t),\ (+4\sin t) \rangle$$

$$r'\left(t = \tfrac{\pi}{2}\right) = \langle 2\sin\left(\tfrac{\pi}{2}\right) + \cancel{2\left(\tfrac{\pi}{2}\right)\cos\left(\tfrac{\pi}{2}\right)}^{\,0},\ 4\sin\left(\tfrac{\pi}{2}\right) \rangle$$

$$= \langle 2, 4 \rangle$$

$$\left|\vec{r}'\left(\tfrac{\pi}{2}\right)\right| = \sqrt{(2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

(length 5) $$r'\left(\tfrac{\pi}{2}\right) = 5 \times \left\langle \frac{2}{2\sqrt{5}}, \frac{\cancel{4}^{\,2}}{\cancel{2}\sqrt{5}} \right\rangle = \boxed{\langle \sqrt{5}, 2\sqrt{5} \rangle}$$ Ans.

Problem 6. Find the velocity, acceleration and speed of the particle with the position function given by $\vec{r}(t) -< 4\cos(2t), 3\sin(2t) >$ at the time $t - \frac{\pi}{3}$.

$$\vec{r}(t) = \langle 4\cos(2t), 3\sin(2t) \rangle$$

$$\vec{v}(t) = \vec{r}'(t) = \langle 4.(-\sin(2t)).(2),\ 3.\cos(2t)(2) \rangle$$

$$= \boxed{\langle -8\sin(2t),\ 6\cos(2t) \rangle}$$

$$\vec{a}(t) = \vec{r}''(t) = \langle (-8).\cos(2t).2,\ 6.(-\sin(2t)).2 \rangle$$

$$= \boxed{\langle -16\cos(2t),\ -12\sin(2t) \rangle}.$$

$$(+\sqrt{3}/2) \qquad \left(-\tfrac{1}{2}\right)$$

Ans: $$\vec{v}\left(t = \tfrac{\pi}{3}\right) = \langle (-8)\sin\left(\tfrac{2\pi}{3}\right), (6)\cos\left(\tfrac{2\pi}{3}\right) \rangle = \langle -4\sqrt{3}, -3 \rangle$$

$$\vec{a}\left(t = \pi/3\right) = \langle (-16)\cos\left(\tfrac{2\pi}{3}\right), (-12)\sin\left(\tfrac{2\pi}{3}\right) \rangle = \langle 8, -6\sqrt{3} \rangle.$$

$$\text{speed} = \left|\vec{v}\left(\tfrac{\pi}{3}\right)\right| = \sqrt{(-4\sqrt{3})^2 + (-3)^2} = \sqrt{16(3) + 9}$$

$$= \sqrt{57}$$