WIR 6

Week-in-Review 6: Chapter 3.1 - 3.4
(Rules of derivatives, Product Rule, Quotient Rule, Chain Rule)

Problem 1. Find the derivatives for the following:

a $f(x) = 3x - 2\sqrt{1-x} + \dfrac{1}{2\sqrt{1-x}}$ $\longrightarrow$ ③ $\frac{1}{2}(1-x)^{-1/2}$

$$f'(x) = 3 - \frac{2}{2\sqrt{1-x}} \cdot \underbrace{\frac{d}{dx}(1-x)}_{(-1)} + \frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{(-1/2-1)} \underbrace{\frac{d}{dx}(1-x)}_{(-1)}$$

$$= 3 + \frac{1}{\sqrt{1-x}} + \frac{1}{4}(1-x)^{-3/2}$$

b $f(x) = x\sin^2 x\cos x$ $= (x)(\sin^2 x)(\cos x)$

$$f'(x) = \cancel{\frac{d}{dx}(x)}^{1} \cdot \sin^2(x) \cdot \cos(x) + \cancel{\frac{d}{dx}(\sin^2 x)}^{2\sin x\cos x}\ x\cos(x) + \cancel{\frac{d}{dx}(\cos x)}^{-\sin x} \cdot x\sin^2 x$$

$$f'(x) = \sin^2(x)\cos(x) + 2x\sin(x)\cos^2(x) - x\sin^3(x)$$

c $f(x) = (\sin x + \cos x)(x^2 - \tan x)$

$$f'(x) = (\cos(x) - \sin(x)) \cdot (x^2 - \tan(x)) + (\sin(x) + \cos(x)) \cdot (2x - \sec^2 x)$$

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\boxed{\frac{d}{dx}(x) = 1}$; $\frac{d}{dx}(-x) = -1$

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x\tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\csc x) = -\csc(x)\cot x$

$$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x)^2$$
$$= 2\sin x^{(2-1)} \cdot \frac{d}{dx}(\sin x)$$
$$= 2\sin(x)\cos(x)$$

$$\frac{d}{dx} \quad = 2\sin(x)\cos(x)$$

2

d $f(x) = \sqrt{x^2+3x+1} - x$

$$f'(x) = \frac{1}{2\sqrt{x^2+3x+1}} \cdot \frac{d}{dx}(x^2+3x+1) - \frac{d}{dx}(x)$$

$$= \frac{2x+3}{2\sqrt{x^2+3x+1}} - 1$$

QR: $\frac{d}{dx}\left(\frac{hi}{lo}\right) = \frac{lo\, dhi - hi\, dlo}{lo \cdot lo}$

e $f(x) = \dfrac{4-x^2}{\sqrt{x}}$ $= \dfrac{4}{\sqrt{x}} - \dfrac{x^2}{\sqrt{x}}$

$$f(x) = \underbrace{4x^{-1/2}} - x^{3/2}$$

$$f'(x) = \cancel{4}^{2}\left(-\frac{1}{\cancel{2}}\right)x^{-3/2} - \left(\frac{3}{2}\right)x^{1/2} = -2x^{-3/2} - \frac{3}{2}x^{1/2}$$

$$= \frac{-2}{x\sqrt{x}} - \frac{3}{2}\sqrt{x}$$

f $f(x) = (x^3-5)^{10}$

$$f'(x) = 10(x^3-5)^{10-1} \cdot \frac{d}{dx}(x^3-5)$$

$$= 10(x^3-5)^9 \cdot (3x^2)$$

$$= 30x^2(x^3-5)^9$$

can use Q.R.

g $f(x) = \frac{(2x+3)^3}{(4x^2+1)^8}$ OR $= \underbrace{(2x+3)^3}_{1} \cdot \underbrace{(4x^2+1)^{-8}}_{2}$ → use P.R

$$f'(x) = (4x^2+1)^{-8} \cdot 3(2x+3)^2 \cdot (2) + (2x+3)^3 \cdot (-8)(4x^2+1)^{-9} \cdot (4 \cdot 2x)$$

$$= \frac{(4x^2+1)}{(4x^2+1)} \frac{6(2x+3)^2}{(4x^2+1)^8} - \frac{64 \cdot x \cdot (2x+3)^3}{(4x^2+1)^9}$$

$$= \frac{6(2x+3)^2(4x^2+1) - 64x(2x+3)^3}{(4x^2+1)^9}$$

h $f(x) = \frac{x^2e^x \,\leftarrow \text{hi}}{x^2+3e^x \,\rightarrow \text{lo}}$

use Q.R.

$$f'(x) = \frac{(x^2+3e^x)(x^2e^x+2xe^x) - (x^2e^x)(2x+3e^x)}{(x^2+3e^x)^2}$$

$$\frac{d}{dx}(x^2 e^x) \xrightarrow{\text{P.R.}} = x^2e^x + 2xe^x$$

i $f(x) = \sqrt{2 + e^x + \sin^2 x}$

$$f'(x) = \frac{1}{2\sqrt{2+e^x+\sin^2 x}} \cdot \frac{d}{dx}\left(\overset{\to 0}{2} + \overset{\to e^x}{e^x} + \overset{\to 2\sin(x)\cos(x)}{\sin^2 x}\right)$$

$$= \frac{e^x + 2\sin(x)\cos(x)}{2\sqrt{2+e^x+\sin^2(x)}}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{-x}) = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

4

j $f(x) = \dfrac{4+x}{xe^x}$ $= \dfrac{4}{xe^x} + \dfrac{x}{xe^x} = \underbrace{4x^{-1}e^{-x}}_{P.R.} + e^{-x}$

$$f'(x) = \underbrace{4(-1)x^{-2}\cdot e^{-x} + 4x^{-1}(-e^{-x})} + (-e^{-x})$$

$$= \frac{-4}{x^2e^x} - \frac{4}{xe^x}\cdot\frac{x}{x} - \frac{1}{e^x}\cdot\frac{x^2}{x^2} = \frac{-4-4x-x^2}{x^2e^x}$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

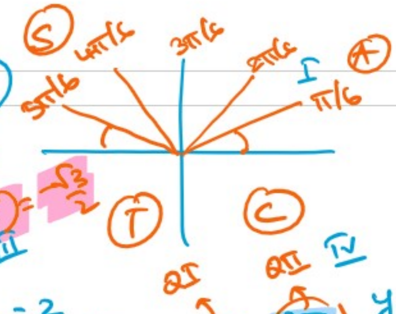
k $f(x) = \tan(\sec(\sin(7x^3)))$ ① ② ③ ④

$$f'(x) = ①\sec^2(\sec(\sin(7x^3)))\cdot\left[②\sec(\sin(7x^3))\cdot\tan(\sin(7x^3))\right]\cdot ③\cos(7x^3)\cdot ④(21x^2)$$

l $f(x) = 2^{e^x}$ $= 2^{(e^x)}$ ← 2 steps.

$$f'(x) = \underbrace{2^{e^x}\cdot\ln(2)}\cdot\underbrace{e^x}$$

Ex: $f(x) = \sin^2(3x^2+5)$ ① ② ③



sin is (+)ve
cos is (−)ve

$\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ but $\cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

If $\cos t = +\frac{\sqrt{3}}{2}$
$t = \pi/6$.

x: $4\sin t = 2$
$\sin t = \frac{2}{4} = \frac{1}{2}$ $\therefore t = \frac{\pi}{6}, \frac{5\pi}{6}$ (QI, QII)

y: $4\cos t = -2\sqrt{3}$
$\cos t = -\sqrt{3}/2$
$t = 5\pi/6$ → in QII

$\cos(-6)$ x: $4\sin t = 2$

$\sin t = \frac{2}{4} = \frac{1}{2}$ $\therefore t = \frac{\pi}{6}, \frac{5\pi}{6}$ y: $\cos t = -\sqrt{3}/2$

$t = 5\pi/6$ → in QII

Problem 2. If $\vec{r}(t) = < 4\sin t, 4\cos t >$ is the position vector of a moving particle at time t, find the velocity and speed of the particle at the point $(2, -2\sqrt{3})$. What is the particle's acceleration at this point?

$\Rightarrow \bar{v}(t) = \bar{r}'(t) = \langle 4\cos t, -4\sin t \rangle$

Pt: $(2, -2\sqrt{3})$, $t = ? = \frac{5\pi}{6}$.

$\bar{v}\left(t = \frac{5\pi}{6}\right) = \left\langle 4\cos\left(\frac{5\pi}{6}\right), -4\sin\left(\frac{5\pi}{6}\right)\right\rangle$

$= \left\langle 4\left(-\frac{\sqrt{3}}{2}\right), -4\left(\frac{1}{2}\right)\right\rangle$

$= \langle -2\sqrt{3}, -2 \rangle$

$\bar{a}(t) = \langle -4\sin t, -4\cos t \rangle$

$\bar{a}\left(\frac{5\pi}{6}\right) = \left\langle (-4)\frac{1}{2}, -4\left(-\frac{\sqrt{3}}{2}\right)\right\rangle$

$\bar{a}\left(\frac{5\pi}{6}\right) = \langle -2, 2\sqrt{3} \rangle$

speed @ $\left(t = \frac{5\pi}{6}\right) = \left|\bar{v}\left(\frac{5\pi}{6}\right)\right| = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$

$= \sqrt{12 + 4} = \sqrt{16}$

$= 4$

Problem 3. Find the equation of the tangent line to the curve $y = x^4 + 2e^x$ at the point $(0, 2)$.

$y = x^4 + 2e^x$

$y' = 4x^3 + 2e^x$

$m = y'|_{x=0} = 0 + 2e^0 = 2$

Pt: $(0, 2)$

$m = y'(x=0)$

Eq$^{\underline{n}}$: $y - 2 = 2(x - 0)$

$$\boxed{y = 2x + 2}$$

Problem 4. Find the equation of the tangent line to the curve $y = 2xe^x$ at the point $(0, 0)$.

$y = 2(x)(e^x)$

Pt: $(0,0)$

$m = y'|_{x=0}$

$y' = 2(x)\cdot(e^x) + 2(e^x)$

$= 2xe^x + 2e^x$

$m = y'|_{x=0} = 2\cdot\cancel{(0)}\cdot e^0 + 2\cdot \cancel{e^0}^{1} = 2$

Eq^n: $y - 0 = 2(x - 0)$

$\boxed{y = 2x}$

Problem 5. Given that $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x, find $f''(2)$.

$f''(x) = \frac{d}{dx}(f'(x))$

Given: $f'(x) = x^2 f(x)$

$f''(x) = (x^2)\cdot f'(x) + (2x) f(x)$

$x=2$

$f''(2) = 2^2 \cdot f'(2) + 2\cdot(2)\cdot f(2)$

$f'(x) = x^2 f(x)$

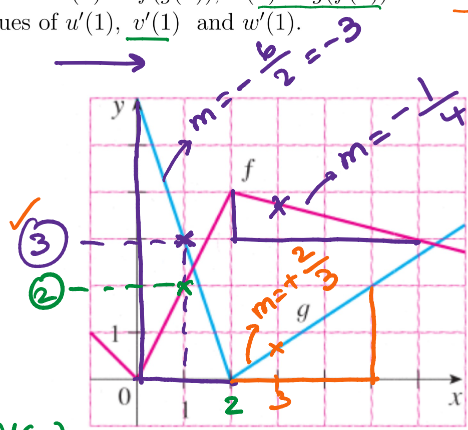
$f'(2) = 2^2 \cdot f(2)$

$f''(2) = (4)\cdot(4)\cdot(10) + (4)(10)$

$= 160 + 40$

$\boxed{f''(2) = 200}$

Problem 6. If we define $u(x) = f(g(x))$, $v(x) = g(f(x))$ and $w(x) = g(g(x))$, use the graph below to find the values of $u'(1)$, $v'(1)$ and $w'(1)$.



$u(x) = f(g(x))$ C.R.

$u'(x) = f'(g(x)) \cdot g'(x)$

$u'(1) = f'(g(1)) \cdot g'(1)$

$= f'(3) \cdot g'(1)$

$= \left(-\frac{1}{4}\right)(-3)$

$\boxed{u'(1) = +\frac{3}{4}}$

$v(x) = g(f(x))$

$v'(x) = g'(f(x)) \cdot f'(x)$

$v'(1) = g'(f(1)) \cdot f'(1)$

$= \underbrace{g'(2)}_{\text{DNE}} \cdot f'(1)$

$g(x=2)$ is a corner

$\therefore g'(x=2)$ DNE

$\therefore \boxed{v'(1) \text{ DNE}}$

$w(x) = g(g(x))$

$w'(x) = g'(g(x)) \cdot g'(x)$

$w'(1) = g'(g(1)) \cdot g'(1)$

$= g'(3) \cdot g'(1)$

$= \left(\frac{2}{\cancel{3}}\right)(-\cancel{3})$

$\boxed{w'(1) = -2}$

Problem 7. Find the equation of the tangent line for $f(x) = 2x\sin(x)$ at the point $(\pi/2, \pi)$.

$f(x) = 2x\sin(x)$ Pt: $(\frac{\pi}{2}, \pi)$

$f'(x) = 2\cdot\sin(x) + 2x\cdot\cos(x)$; $m = f'(x = \frac{\pi}{2})$

$m = f'(\frac{\pi}{2}) = 2\cdot\overset{1}{\cancel{\sin}}(\frac{\pi}{2}) + 2\cdot\frac{\pi}{2}\cdot\overset{0}{\cancel{\cos}}(\frac{\pi}{2})$

$m = 2.$

Eqn : $\boxed{y - \pi = 2\left(x - \frac{\pi}{2}\right)}$

Problem 8. If $g(x) = (2 - x^2)^6$, find $g(0), g'(0), g''(0)$ and $g^3(0)$.

0. $g(x) = (2-x^2)^6$; $g(0) = (2-0)^6 = 2^6$

1. $g'(x) = 6(2-x^2)^5\cdot(-2x)$
$= (-12x)(2-x^2)^5$; $g'(0) = 0$

2. $g''(x) = (-12)(2-x^2)^5 - (12x)5\cdot(2-x^2)^4(-2x)$
$= -12(2-x^2)^5 + \underbrace{120(x^2)(2-x^2)^4}$
$g''(0) = -12(2^5)$

3. $g'''(x) = (-12)\cdot 5(2-x^2)^4(-2x)$
$+ (120)(2x)(2-x^2)^4$
$+ (120x^2)\cdot 4(2-x^2)^3(-2x)$ $g'''(0) = 0.$

$\frac{d}{dx}(1-u) = -1$

$\frac{d}{dx}(1-u) = -1$

Problem 9. Find the second derivatives for the following

a $f(u) = \frac{1}{\sqrt{1-u}}$. $= (1-u)^{-1/2}$

$f'(u) = \left(-\frac{1}{2}\right)(1-u)^{-3/2} \cdot (-1) = \frac{1}{2}(1-u)^{-3/2}$

$f''(u) = \left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1-u)^{-5/2}(-1)$

$= +\frac{3}{4}(1-u)^{-5/2}$

b $f(x) = e^x - 5x^2$.

$f'(x) = e^x - 5(2x) = e^x - 10x$

$f''(x) = e^x - 10$

notice $\begin{cases} f^3(x) = e^x \\ f^4(x) = e^x \end{cases}$ $f^n(x) = e^x$, $n \geq 3$

c $f(t) = \underbrace{(t^3+1)}\underbrace{e^t}$ $\to$ P.R

$f'(t) = \underbrace{(t^3+1)\cdot(e^t)} + \underbrace{(3t^2)(e^t)}$

$f''(t) = \underbrace{(t^3+1)(e^t) + (3t^2)(e^t)} +$

$\underbrace{3(t^2)(e^t) + 3(2t)\cdot(e^t)}$

tangent line is slope m
horizontal slope $\Rightarrow m=0$.

Problem 10. Find all the values of x where the tangent line to the function $f(x) = 2\sin x + \sin^2 x$ is horizontal.

Ans: $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\boxed{x = (2n+1)\frac{\pi}{2} \quad n \geq 0}$

$f(x) = 2\sin(x) + \sin^2(x)$

$m = f'(x) = 2\cos(x) + 2\sin(x)\cos(x)$

when $m=0$

$2\cos(x) + 2\sin(x)\cos(x) = 0$

$\underbrace{2\cos(x)}_{\cos(x)=0}\big(\underbrace{1+\sin(x)}_{1+\sin(x)=0}\big) = 0.$

$\cos(x)=0$
when $x = \pi/2, 3\pi/2$

$1+\sin(x)=0$
$\sin(x)=-1$
when $x = 3\pi/2$

Problem 11. Given that $f(x) = x\sin(x)$, find the 35^{th} derivative of $f(x)$.

$f(x) = x\sin(x)$

$n=35$
$=(4\times 8)+3$

$n=1 \quad f'(x) = \sin(x) + x\cos(x)$

$n=2 \quad f^2(x) = \cos(x) + \cos(x) - x\sin(x)$
$= 2\cos(x) - x\sin(x)$

$\Rightarrow n=3 \quad f^3(x) = -2\sin(x) - \sin(x) - x\cos(x)$
$= -3\sin(x) - x\cos(x)$

$n=4 \quad f^4(x) = -3\cos(x) - \cos(x) + x\sin(x)$
$= -4\cos(x) + x\sin(x)$

$$\boxed{f^{35}(x) = -35\sin(x) - x\cos(x)}$$

Problem 12. Find the n^{th} derivative for the function $y = \dfrac{1}{x^2}$

$$y = \frac{1}{x^2} = x^{-2}$$

$n=1$ $\quad y' = (-2)x^{-3}$

$n=2$ $\quad y'' = (-2)(-3)x^{-4}$

$n=3$ $\quad y^3 = (-2)(-3)(-4)x^{-5}$

$$y^4 = (-2)(-3)(-4)(-5)x^{-6}$$

$$y^5 = (-2)(-3)(-4)(-5)(-6)x^{-7}$$

$(1+1)$, $(1+2)$, $(2+2)$, $(3+2)$

$$y^n = (-1)^n \underbrace{(2\cdot 3\cdot 4\cdot \ldots \cdot n\cdot(n+1))}_{(n+1)!}x^{-(n+2)}$$

nth derivative

$$\boxed{y^n = \frac{(-1)^n\cdot(n+1)!}{x^{(n+2)}}}$$