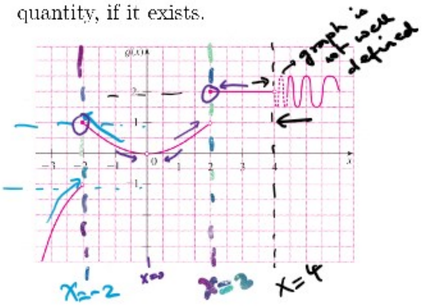
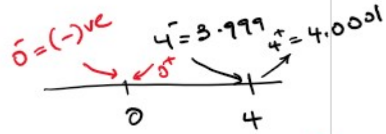
WIR 3

Week-in-Review 3
(Limits (2.2), Limit Laws (2.3) and Continuity (2.5))

Problem 1. For the function $g(x)$ whose graph is given below, state the value of the given quantity, if it exists.



a) $\lim_{x\to -2} g(x)$.
b) $g(-2)$
c) $\lim_{x\to 2} g(x)$.
d) $g(2)$.
e) $\lim_{x\to 0} g(x)$, $g(0)$.
f) $\lim_{x\to 4^-} g(x)$, $\lim_{x\to 4^+} g(x)$.



In order for $\lim_{x\to a} f(x) = L$

LHL = RHL

a) $\boxed{\lim_{x\to(-2)} g(x)} \to$ DNE

LHL $= -1 \neq$ RHL $= 1$

b) $g(-2) = 1$

c) $\lim_{x\to 2} g(x) \to$ DNE

LHL $= +1 \neq$ RHL $= +2$

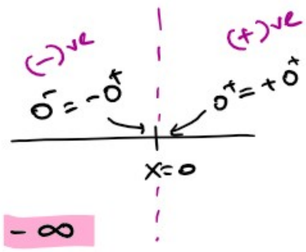
d) $g(2) = 2$

e) $\lim_{x\to 0} g(x) = 0$ (LHL = RHL = 0) | $g(0)$ DNE

f) $\lim_{x\to 4^-} g(x) = 2$ (LHL) | $\lim_{x\to 4^+} g(x)$ DNE

$+\infty$ or $-\infty$

Problem 2. Find the infinite limit for the following:



(a) $\lim_{x\to 0^-} \frac{x-1}{x^2(x+2)}$.

LHL

$$\sim \frac{-0^+ - 1}{(-0^+)^2(-0^+ + 2)}$$

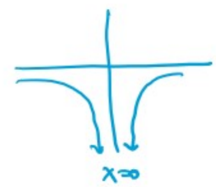
$$= \frac{-1}{(+0^+)(2)} \sim -\frac{1}{0^+} \to -\infty$$

(b) $\lim_{x\to 0^+} \frac{x-1}{x^2(x+2)}$.

RHL

$$\sim \frac{0^+ - 1}{(0^+)^2(0^+ + 2)} \sim \frac{-1}{(0^+)(2)} \to -\infty$$

$$\lim_{x\to 0} \frac{(x-1)}{(x^2)(x+2)} \to -\infty$$



(c) $\lim_{x\to 0^-} \frac{x-1}{x(x-2)}$.

LHL

$$\sim \frac{-0^+ - 1}{(-0^+)(-0^+ + 2)} \sim \frac{\left(-\frac{1}{2}\right)}{(-0.000001)} = +\infty$$

$$\sim \frac{-1}{(-0^+)(2)} \sim \frac{1}{(0^+)(2)} \to +\infty$$

(d) $\lim_{x\to 0^+} \frac{x-1}{x(x-2)}$.

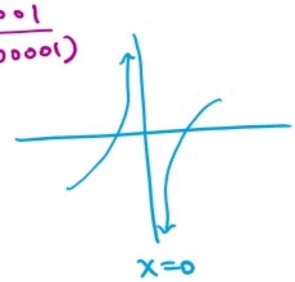
RHL

$$\sim \frac{0^+ - 1}{(0^+)(0^+ + 2)} \sim \frac{-1}{(0^+)(2)} \to -\infty$$

$$\frac{-1.000001}{(0.000001)(2.00001)}$$

LHL $\neq$ RHL

$$\lim_{x\to 0} \frac{(x-1)}{x(x+2)} \quad \text{DNE}$$



a) & b) $\sim \frac{1}{x^2}$ vs c) & d) $\sim \frac{1}{x}$

Problem 3. Given that $\lim_{x\to a} f(x) = -3$, $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} h(x) = 8$, find the limits that exist.

(a) $\lim_{x\to a}[f(x) + h(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} h(x) = -3 + 8 = 5$

(a) $\lim_{x\to a}[f(x)+h(x)]$ $= \lim_{x\to a} f(x) + \lim_{x\to a} h(x) = -3+8 = 5$

(b) $\lim_{x\to a}[f(x)]^2$ $= \left(\lim_{x\to a} f(x)\right)^2 = (-3)^2 = 9$

(c) $\lim_{x\to a}\sqrt[3]{h(x)}$ $= \sqrt[3]{\lim_{x\to a} h(x)} = \sqrt[3]{8} = +2$

(d) $\lim_{x\to a}\frac{1}{f(x)}$ $= \frac{1}{\lim_{x\to a} f(x)} = \frac{1}{-3} = -\frac{1}{3}$

(e) $\lim_{x\to a}\frac{f(x)}{h(x)}$ $= -\frac{3}{8}$

(f) $\lim_{x\to a}\frac{g(x)}{f(x)}$ $= \frac{0}{-3} = 0$

(g) $\lim_{x\to a}\frac{f(x)}{g(x)}$ $= \frac{-3}{0} \rightarrow$ DNE

(h) $\lim_{x\to a}\frac{2f(x)}{h(x)-f(x)}$ $= \frac{2(-3)}{8-(-3)} = \frac{-6}{8+3} = -\frac{6}{11}$

2^3

$$\underset{(2+x)}{a^3} - \underset{(2)}{b^3} = (a-b)(a^2+ab+b^2)$$

Problem 4. Evaluate the fllowing limts if they exist:

(a) $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$ $\sim \frac{(2+0)^3-8}{0} = \frac{8-8}{0} \sim \frac{0}{0}$

$$= \frac{(\cancel{2}+x-\cancel{2})\left((2+x)^2+(2+x)(2)+(2)^2\right)}{x}$$

$$= \lim_{x\to 0} \frac{\cancel{x}(4+4x+x^2+4+2x+4)}{\cancel{x}}$$

$$= 4+4\cancel{(0)}+\cancel{(0)^2}+4+2\cancel{(0)}+4 = 12 \quad \text{Ans.}$$

(b) $\lim_{x\to 1} \frac{x^2-x-2}{x^2-3x+2}$ $\sim \frac{1+1-2}{1-3+2} \sim \frac{0}{0}$

$$\lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{(x-2)\cancel{(x-1)}} = \frac{1+2}{1-2} = \frac{3}{-1} = -3$$

-3

$$a^2-b^2 = (a+b)(a-b)$$

(c) $\lim_{x\to 1}[\frac{1}{x-1} - \frac{2}{x^2-1}]$ $\sim \frac{1}{1-1} - \frac{2}{1-1} \sim \frac{1}{0} - \frac{2}{0} \sim \infty - \infty$

$\neq 0$

$$\frac{(x+1)}{(x+1)}\frac{1}{(x-1)} - \frac{2}{(x+1)(x-1)} = \frac{(x+1)-2}{(x+1)(x-1)}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \frac{1}{2} \quad \text{Ans.}$$

absolute value fn

(d) $\lim_{x\to 1.5} \frac{2x^2-3x}{|2x-3|}$ → DNE

$2x-3=0$
$x=\frac{3}{2}=1.5$

$(-2x-3)=(3-2x)$

$|2x-3| = \begin{cases} -(2x-3), & x<1.5 \to \text{LHL} \\ (2x-3), & x \geq 1.5 \to \text{RHL} \end{cases}$

$\sim \frac{2\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)-3\left(\frac{3}{2}\right)}{\left|2\cdot\frac{3}{2}-3\right|} \sim \frac{0}{0}$

LHL → ← RHL
$x=1.5$

$x\to 1.5^-$

LHL

$\lim_{x\to 1.5} \frac{x(\cancel{2x-3})}{-(\cancel{2x-3})} = -1.5$

$\neq$

RHL

$\lim_{x\to 1.5} \frac{x(\cancel{2x-3})}{(\cancel{2x-3})} = 1.5$

LHL (e) $\lim_{x\to 2^-} \sqrt{x^2+x-2}$

$\sim \sqrt{(2^-)^2+(2^-)-2}$

$= \sqrt{4+(2^- -2)}$

$= \sqrt{4-0^+}$

$= \sqrt{4^-}$

~ 2

2^-
$x=0$ $\quad$ $x=2$

Imp. $\sqrt{4}=+2$ only

4^-
$x=0$ $\quad$ $x=4$

$4-4^- = +0^+$

$\lim_{x\to -2^-} \sqrt{x^2+x-2} \sim \sqrt{(-2^-)^2+(-2^-)-2}$

$= \sqrt{4-4^-}$

$= \sqrt{+0^+} \sim 0$

If $\sqrt{-0^-}$ → DNE

Problem 5. What is wrong with the following equation?

$$\frac{x^2+x-6}{x-2} \neq x+3$$

$f(x)$ $\qquad$ $g(x)$

D: $x\neq 2$ $\qquad$ D : all x

$\frac{\quad}{x-2} \neq x+3$

D: $x \neq 2$ D: all x

LHS $\frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}}$

$f(2)$ DNE

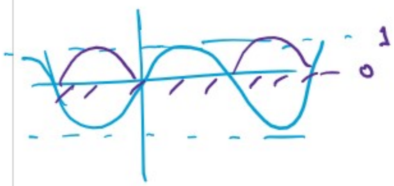
$\lim_{x\to 2} f(x) = g(x)$

$$\lim_{x\to 2}\left(\frac{x^2+x-6}{x-2}\right) = (x+3)$$

Problem 6. Use the Squeeze theorem to show that $\lim_{x\to 0}\sqrt{x^3+x^2}\sin(\frac{\pi}{x}) = 0$

$$\lim_{x\to 0}\sqrt{x^3+x^2}\cdot\sin\left(\frac{\pi}{x}\right) \sim \left(\sqrt{0^3+0^2}\right)\sin\left(\frac{\pi}{0}\right)$$

$$\sim (0)(\sin\infty)$$



$$\boxed{-1 \le \sin(x) \le +1}$$

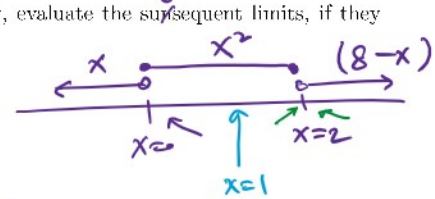
also $-1 \le \cos(x) \le 1$

$0 \le \sin^2(x) \le 1$

$$\lim_{x\to 0}\left(\sqrt{\cancel{x^3}^{\,0}+x^2}\right)(-1) \le \lim_{x\to 0}\left(\sqrt{x^3+x^2}\right)\sin\left(\frac{\pi}{x}\right) \le \lim_{x\to 0}(+1)\left(\sqrt{\cancel{x^3}^{\,0}+x^2}\right)$$

$$0 \le \underbrace{\lim_{x\to 0}\left(\sqrt{x^3+x^2}\right)\left(\sin\left(\frac{\pi}{x}\right)\right)}_{\text{must be zero!}} \le 0$$

Problem 7. Given the piecewise function $h(x)$ below, evaluate the subsequent limits, if they exist.



$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 2 \\ 8-x & \text{if } x > 2 \end{cases}$$

RHL (a) $\lim_{x\to 0^+} h(x)$. $= \lim_{x\to 0^+}(x^2) = 0$

(b) $\lim_{x\to 0} h(x)$. $= 0$

LHL: $\lim_{x\to 0^-}(x) = 0$ Since LHL = RHL = 0

(c) $\lim h(x)$. $= \lim (x^2) = 4$

LHL : $\lim_{x \to 0^-} (x) = 0$ since LHL ≠ RHL

(c) $\lim_{x \to 1} h(x)$. $= \lim_{x \to 1} (x^2) = 1$

(d) $\lim_{x \to 2^-} h(x)$. $= \lim_{x \to 2^-} (x^2) = 4$

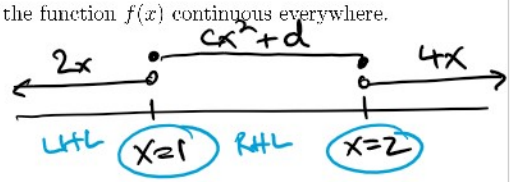
(e) $\lim_{x \to 2^+} h(x)$. $= \lim_{x \to 2^+} (8-x) = 8-2 = 6$

dont agree

(f) $\lim_{x \to 2} h(x)$. DNE

Problem 8. Find the values of c and d that make the function $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ cx^2 + d & \text{if } 1 \le x \le 2 \\ 4x & \text{if } x > 2 \end{cases}$$



@ x=1

LHL

$\lim_{x\to 1^-} (2x) = 2$ $=$

RHL

$\lim_{x\to 1^+} (cx^2+d) = c+d$

$= f(1)$
$= c+d.$
$= \text{RHL}$

$c+d=2 \rightarrow d = 2-c$

@ x=2

LHL

$\lim_{x\to 2^-} (cx^2+d)$
$= 4c+d$

$=$

RHL

$\lim_{x\to 2^+} (4x) = 8$

$= f(2)$
$4c+d.$
$= \text{LHL}$

$4c+d=8$

$4c + (2-c) = 8$

$4c - c = 8-2$

$3c = 6$

$\therefore c=2$

$d = 2-c = 2-2=0.$

$d=0$

Problem 9. Which of the following functions $f(x)$ has a removable discontinuity at $x = a$?

hole

RD

(a) $f(x) = \dfrac{x^2-2x-8}{x+2}$, $a = -2$

$f(x) = (x-4)\cancel{(x+2)}$ $\therefore$ $x=-2$ is a hole

(a) $f(x) = \frac{}{x+2}$, $a = -2$

$$f(x) = \frac{(x-4)\cancel{(x+2)}}{\cancel{(x+2)}}$$

$\therefore$ $x=-2$ is a hole

Ans: $f(x)$ does have a RD

(b) $f(x) = \frac{x-7}{|x-7|}$, $a = 7$

$$|x-7| = \begin{cases} -(x-7), & x < 7 \\ (x-7), & x \geq 7 \end{cases}$$

LHL $\qquad\qquad\qquad$ RHL

$$\frac{x-7}{-(x-7)} = -1 \neq \frac{x-7}{+(x-7)} = +1$$

$\lim_{x\to 7}$ DNE

non removable.

(c) $f(x) = \frac{x^3-64}{x+4}$, $a = -4$

(d) $f(x) = \frac{3-\sqrt{x}}{9-x}$, $a = 9$ $\rightarrow \lim_{x\to 9}$. $\sim \frac{3-\sqrt{9}}{9-9} \sim \frac{0}{0}$

$(a-b)(a+b) = a^2-b^2$

$3-\sqrt{x} \xrightarrow{\text{conjugate}} 3+\sqrt{x}$

$$\frac{(3-\sqrt{x})}{(9-x)} \cdot \frac{(3+\sqrt{x})}{(3+\sqrt{x})} = \lim_{x\to 9} \frac{\cancel{(9-x)}}{\cancel{(9-x)}(3+\sqrt{x})} = \frac{1}{3+\sqrt{9}}$$

$$= \frac{1}{6} \neq f(9)$$

Does $a=9$ represent a hole? Yes

Yes it is a R.D.

Problem 10. If $f(x) = x^3 - x^2 - 10$, show that a number c exists such that $f(c) = 10$.

IVT

$\rightarrow$ y value of 10

$f(0) = -10$

$f(1) = 1-1-10 = -10$

$f(2) = 8-4-10 = -6$

$f(3) = 27-9-10 = 8$

$f(4) = 64-16-10 = 38$

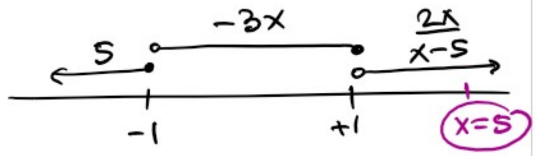
for $f(c)=10$

Ans: $3 \leq c \leq 4$

Problem 11. At what values of x is the following piece-wise function $f(x)$ discontinuous?

Problem 11. At what values of x is the following piece-wise function $f(x)$ discontinuous?

$$f(x) = \begin{cases} 5 & \text{if } x \le -1 \\ -3x & \text{if } -1 < x \le 1 \\ \dfrac{2x}{x-5} & \text{if } x > 1 \end{cases}$$



Ans: @ $x = -1, 1, 5$

$\boxed{\text{VA @ } x = 5}$

@ $x = -1$

LHL (5) $\neq$ RHL $(-3x = -3(-1) = 3)$

discontinuity @ $x = -1$

@ $x = +1$

LHL $(-3x = 3(1) = -3)$ $\neq$ RHL $\frac{2x}{x-5} = \frac{2}{1-5} = \frac{2}{-4} = -\frac{1}{2}$

$\therefore$ discontinuity @ $x = +1$

Problem 12. Test the following functions for continuity. If the function is discontinuous for some value $x = a$, explain why.

(a) $f(x) = \dfrac{x^2-1}{x+1}$, $a = -1$

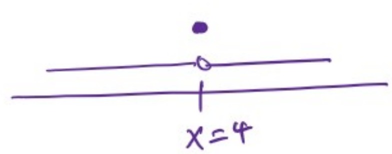
Rem: (x^2-1) is like (a^2-b^2)
$= (a+b)(a-b)$

$$\lim_{x \to (-1)} \left(\frac{x^2-1}{x+1} \right) \sim \frac{0}{0}$$

$$= \lim_{x \to (-1)} \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}} = -1-1 = \boxed{-2}$$

$\therefore$ f^n is discontinuous @ $x = -1$.

(b) $f(x) = \begin{cases} 3 & \text{if } x = 4 \\ \dfrac{x^2-2x-8}{x-4} & \text{if } x \neq 4 \end{cases}$, at $a = 4$



$$\frac{x^2-2x-8}{x-4} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}} = 4+2 = 6.$$

$$\frac{\quad}{x-4} \qquad = \lim_{x \to 4} \frac{\quad}{(x-\cancel{4})}$$

$\downarrow$

discont @ $x=4$

$\boxed{\text{but } x=4 \text{ outside Domain}}$ & $f(4)=3$

of this part of $f(x)$

Ans: $\therefore$ for b) $f(x)$ has no discontinuities