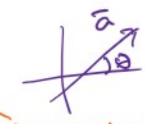


Week-in-Review 2 (Vectors: part 2 and 3)

$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta = a_1b_1 + a_2b_2$

Problem 1. Find the dot product for the following pair of vectors. Are the vectors parallel, perpendicular or neither? If neither, find the angle between the vectors.

↳ info on slope m

$m = \tan\theta = \frac{y_{comp}}{x_{comp}}$

(1) $< 6, 0 >$ and $< 5, 3 >$

$\vec{a}$ $\vec{b}$

$\bar{a}\cdot\bar{b} = \langle 6,0\rangle\cdot\langle 5,3\rangle = (6)(5)+(0)(3) = 30$

$m(\bar{a}) = \frac{0}{6} = 0$, $m(\bar{b}) = \frac{3}{5}$ } neither $\perp^r$ nor parallel

$\cos\theta = \frac{\langle 6,0\rangle\langle 5,3\rangle}{|\langle 6,0\rangle||\langle 5,3\rangle|}$

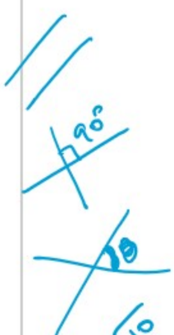
$\therefore \theta = \arccos\left(\frac{\cancel{30}^{5}}{\cancel{(6)}(\sqrt{34})}\right) = 31^\circ$

(2) $< 2, -1 >$ and $< -4, 2 >$

$\vec{a}$ $\vec{b}$

$\bar{a}\cdot\bar{b} = \langle 2,-1\rangle\langle -4,2\rangle = (2)(-4)+(-1)(2) = -10$

$m(\vec{a}) = -\frac{1}{2}$, $m(\vec{b}) = \frac{2}{-4} = -\frac{1}{2}$ } vectors are parallel



(3) $< 6, 2 >$ and $< 1, 3 >$

$\vec{a}$ $\vec{b}$

$\bar{a}\cdot\bar{b} = \langle 6,2\rangle\langle 1,3\rangle = (6)(1)+(2)(3) = 12$

$m(\vec{a}) = \frac{2}{6} = \frac{1}{3}$, $m(\vec{b}) = \frac{3}{1} = 3$ } vectors are neither // nor $\perp^r$

$\theta = \arccos\left(\frac{12}{\sqrt{40}\sqrt{10}}\right)$

$\sqrt{4(10)}\sqrt{10}$

$\sqrt{4}\sqrt{10}\sqrt{10}$

$= 2\times 10$

$\boxed{\theta = \arccos\left(\frac{12}{20}\right)}$ Ans.

$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$ θ = angle between $\bar{a}$ & $\bar{b}$

If $\theta = 90^\circ$, $\cos(90^\circ) = 0 \Rightarrow \bar{a}\cdot\bar{b} = 0$

Problem 2. What value(s) of x will make the vectors $< x, x >$ and $< 1, x >$ orthogonal? → also means perpendicular.

$\vec{a}$ $\vec{b}$

$m_1(\bar{a}) = \frac{x}{x} = 1$

$m_2(\bar{b}) = \frac{x}{1} = x$

$\langle x, x\rangle \langle 1, x\rangle = 0$ for $\perp$ vectors

$x + x^2 = 0$

$x(1+x) = 0$

$x = 0 \rightarrow$ null vector $\vec{a}$

$x = -1$ Ans.

For $\bar{a}$ & $\bar{b}$ to be perpendicular $\boxed{(m_1)(m_2) = -1}$

$(1)(x) = -1$

$x = -1$ Ans.

has length 1. $\perp$

Problem 3. Find the unit vector(s) orthogonal to the vector $< 3, 1 >$.

$\vec{a} \rightarrow |\bar{a}| = \sqrt{10}$

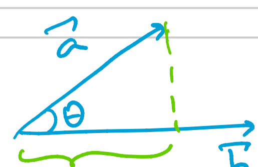
Orthogonal complement: $\langle -1, 3\rangle = \vec{a}^{\perp}$

unit vector $(\bar{a}^{\perp}) = \frac{\langle -1, 3\rangle}{\sqrt{10}}$ Ans. $\rightarrow |\bar{a}^{\perp}| = \sqrt{10}$

Find a unit vector parallel to the vector $< 3, 1 >$.

$\vec{a}$

$\frac{\bar{a}}{|\bar{a}|} = \frac{\langle 3, 1\rangle}{\sqrt{10}}$ Ans.



Scalar: $\text{comp}_{\bar{b}}\bar{a} = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$

vector Vector: $\text{proj}_{\bar{b}}\bar{a} = (\text{comp}_{\bar{b}}\bar{a})\cdot\left(\frac{\bar{b}}{|\bar{b}|}\right)$ scalar projection

Problem 4. Find the scalar projection of $< 3, -1 >$ onto $< 2, 3 >$.

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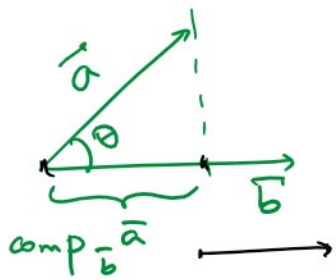
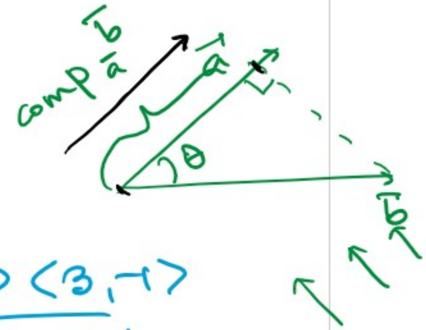
$\vec{a}$ $\vec{b}$

$$\text{comp}_{\bar{b}}\,\bar{a} = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{\langle 3,-1\rangle\langle 2,3\rangle}{|\langle 2,3\rangle|}$$

$$= \frac{6+(-3)}{\sqrt{13}} = \frac{3}{\sqrt{13}} \quad \text{Ans.}$$





Find the vector projection of $< 2, 3 >$ onto $< 3, -1 >$.

$\vec{a}$ $\vec{b}$

$$\text{comp}_{\bar{b}}\,\bar{a} = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{\langle 2,3\rangle\langle 3,-1\rangle}{|\langle 3,-1\rangle|}$$

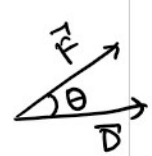
$$= \frac{3}{\sqrt{10}}$$

$$\text{proj}_{\bar{b}}\,\bar{a} = \left(\text{comp}_{\bar{b}}\,\bar{a}\right)\left(\frac{\bar{b}}{|\bar{b}|}\right) = \frac{(\bar{a}\cdot\bar{b})\,\bar{b}}{|\bar{b}|^2}$$

$$= \left(\frac{3}{\sqrt{10}}\right)\left(\frac{\langle 3,-1\rangle}{\sqrt{10}}\right)$$

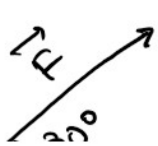
$$= \left(\frac{3}{10}\right)\langle 3,-1\rangle$$

$$W = \vec{F}\cdot\vec{D} = |\bar{F}||\bar{D}|\cos\theta$$



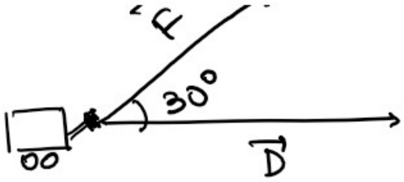
metric

Problem 5. A wagon is pulled a distance of 100 meters along a horizontal path by a constant force of 50 N. If the handle of the wagon is at an angle of 30° above the horizontal, how much work is done?



$|\vec{D}| = 100\text{ m}$

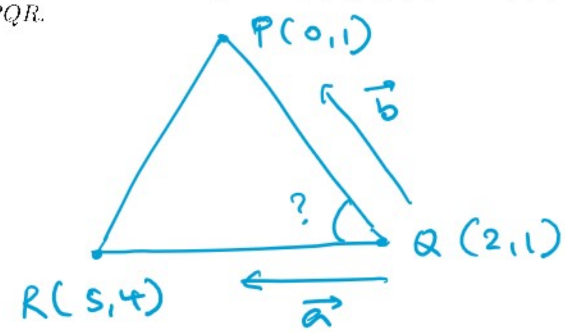
$|\bar{F}| = 50\text{ N}$



$|\vec{F}| = 50$ N

$$W = |\vec{F}||\vec{D}|\cos\theta$$
$$= (50)(100)\cos(30^\circ)$$
$$= \overset{25}{\cancel{50}}(100)\left(\frac{\sqrt{3}}{\cancel{2}}\right)$$
$$= 2500\sqrt{3} \text{ N-m or J}$$

Problem 6. Given that the points $P(0,1), Q(2,1)$ and $R(5,4)$ make the 3 vertices of a triangle, find $\angle PQR$.



angle → dot product

$$\vec{a} = \vec{QR} = \langle (5,4)-(2,1) \rangle$$
$$= \langle 3,3 \rangle$$
$$\vec{b} = \vec{QP} = \langle (0,1)-(2,1) \rangle$$
$$= \langle -2,0 \rangle$$

$$\theta = \arccos\left(\frac{\langle 3,3 \rangle \langle -2,0 \rangle}{|\langle 3,3 \rangle||\langle -2,0 \rangle|}\right) = \arccos\left(\frac{-6+0}{\sqrt{18}\sqrt{4}}\right)$$

$\sim 14^\circ$

$y - y_1 = m(x - x_1)$

Vector Eq^n of a line: $\boxed{\bar{r}(t) = \bar{r}_0 + t\bar{v}}$ (point; slope)

t = parameter | $\vec{r_0}$ → point P_0 on given line to origin | $\bar{v}$ = direction of line.

Problem 7. Find the vector equation of the line $2y + 3x = 5$

$$2y + 3x = 5$$
$$2y = -3x + 5$$
$$y = \left(-\frac{3}{2}\right)x + \frac{5}{2}$$

$m = -\frac{3}{2}$ ⟶ $\vec{v} = \langle 2, -3 \rangle$ or $\langle -2, 3 \rangle$

$x = 1$ $(1,1)$ P_0

$\bar{r}(t) = \langle 1,1 \rangle + t\langle 2,-3 \rangle$

Pt on this line: $(x=0, y=\frac{5}{2})$
⇓
✓P_0 $(0, \frac{5}{2})$

$$\vec{r_0} = \langle 0, \frac{5}{2} \rangle$$

one possible

$\langle -2, 3 \rangle$

$$\boxed{\bar{r}(t) = \left\langle 0, \tfrac{5}{2} \right\rangle + t\langle 2, -3 \rangle}$$ One possible answer.

Problem 8. Find the vector equation of the line that makes an angle of 60° with the positive $x-$axis and passes through the point $(2, -5)$.

θ

$P_0(2, -5) \longrightarrow \vec{r_0} = \langle 2, -5 \rangle$

$\vec{v} \longrightarrow m = \tan\theta = \tan(60^\circ) = \sqrt{3} = \frac{\sqrt{3}}{1}$

$\bar{v} = \langle 1, \sqrt{3} \rangle$

$\bar{r}(t) = \bar{r_0} + t\bar{v}$

$$\boxed{\bar{r}(t) = \langle 2, -5 \rangle + t\langle 1, \sqrt{3} \rangle}$$ One possible solution.

$= \langle 2, -5 \rangle + \langle t, t\sqrt{3} \rangle$

$\vec{r}(t) = \langle \underbrace{(2+t)}_{x}, \underbrace{(-5+t\sqrt{3})}_{y} \rangle$

P_0

Problem 9. Find the Vector Equation of a line that passes through the point $(1, 3)$ and is parallel to the vector $< 1, -2 >$.

$\vec{v}$

$\bar{r}(t) = \vec{r_0} + t\vec{v}$

$$\boxed{\bar{r}(t) = \langle 1, 3 \rangle + t\langle 1, -2 \rangle}$$ Ans.

$= \langle 1, 3 \rangle + \langle t, -2t \rangle$

$\vec{r}(t) = \langle \underbrace{(1+t)}_{x}, \underbrace{(3-2t)}_{y} \rangle$

Find the Parametric Equations for this line.

$\left. \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right\}$

$x = 1 + t$

$y = 3 - 2t$

Ans.

$y = 3-2t$

Eliminate the parameter to find the Cartesian Equation of the line.

$y = f(x)$

$\to y = mx+b$.

$x = 1+t$

$t = x-1$

$y = 3-2t$

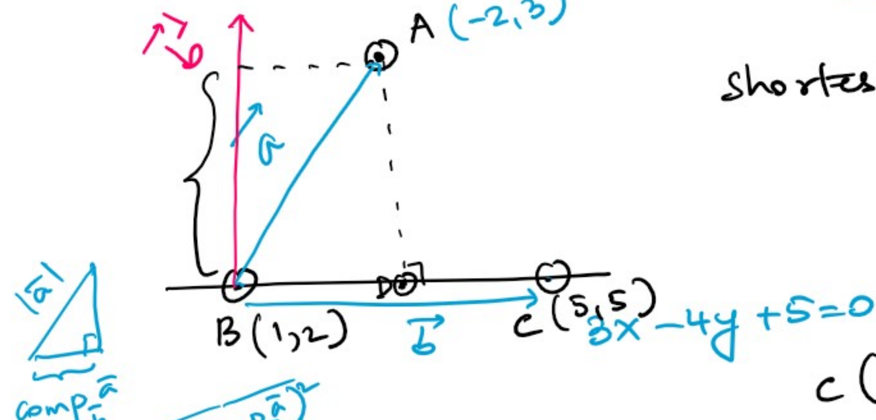
$= 3-2(x-1)$

$= 3-2x+2$

$\boxed{y = -2x+5}$ Ans.

shortest A line

Problem 10. Find the distance of the point $(-2, 3)$ from the line $3x - 4y + 5 = 0$.



shortest distance = length AD → scalar projection

$AD = \sqrt{|a|^2 - (\text{comp}_{\bar{b}}\bar{a})^2}$

$C(x=5, y=5)$

$B(x=1, y=2)$

$\vec{a} = \vec{BA} = \langle(-2,3)-(1,2)\rangle = \langle -3, 1\rangle$

$\vec{b} = \vec{BC} = \langle(5,5)-(1,2)\rangle = \langle 4, 3\rangle$

$\vec{b}^{\perp} = \langle -3, 4\rangle$

$|\vec{b}^{\perp}| = \sqrt{9+16} = \sqrt{25} = 5$

$$\left|\text{comp}_{\vec{b}^{\perp}}\vec{a}\right| = \left|\frac{\vec{a}\cdot\vec{b}^{\perp}}{|\vec{b}^{\perp}|}\right|$$

$$= \left|\frac{\langle -3, 1\rangle\langle -3, 4\rangle}{\sqrt{25}}\right|$$

$$= \left|\frac{(-3)(-3)+(1)(4)}{5}\right| = \frac{13}{5}$$

Ans. units

Problem 11. The position of an object moving in the xy-plane, after t seconds, is given by

Problem 11. The position of an object moving in the $xy-$plane, after t seconds, is given by $\vec{r}(t) = (t+4)\vec{i} + (t^2+2)\vec{j}$.

(1) Find the position of the object at time $t = 2$.

$x = t+4$
$y = t^2+2$

$\vec{r}(2) = 6\hat{i} + 6\hat{j} = \langle 6, 6 \rangle$

$\longrightarrow$ Point $(6,6)$

(2) At what time will the object reach the point $(7, 11)$?

$x = 7 \;,\; y = 11$

$x = t+4 = 7 \Rightarrow t = 3$

$y = t^2+2 = 11 \Rightarrow t^2 = 9 \;,\; t = 3$

Ans: @ $t = 3$ seconds.

(3) At what time will the object reach the point $(9, 12)$?

$x = 9, \; y = 12$

$x = t+4 = 9 \Rightarrow t = 5$

$y = t^2+2 = 12 \Rightarrow t^2 = 10$

$t = \sqrt{10}$

t does not agree for x & y

$\therefore$ Object will never be @ $(9,12)$

(4) Find the Cartesian equation describing the path of the object.

$x = t+4 \longrightarrow t = x-4$

$y = t^2+2$

$\therefore \boxed{y = (x-4)^2+2}$ Ans.

$\sim \; y = f(x)$

no t in it

Problem 12. Are the following pairs of lines parallel, perpendicular or neither? If the lines are not parallel, find the point of intersection between them.
$\vec{r_1}(t) = (-4+2t)\vec{i} - (5+t)\vec{j}$ and $\vec{r_2}(s) = (2+3s)\vec{i} + (4-6s)\vec{j}$.

$\vec{r}(t) = \vec{r_0} + t\vec{v}$

$$\boxed{\vec{r}_1(t)} = \langle (-4+2t), (5+t) \rangle$$
$$= \langle -4, 5 \rangle + \langle 2t, t \rangle$$
$$= \underbrace{\langle -4, 5 \rangle}_{\vec{r_0}} + t \underbrace{\langle 2, 1 \rangle}_{\vec{v}} \qquad m_1 = \frac{1}{2}$$

$$\vec{r}_2(s) = \langle (2+3s), (4-6s) \rangle$$
$$= \langle 2, 4 \rangle + \langle 3s, -6s \rangle$$
$$= \langle 2, 4 \rangle + s\langle 3, -6 \rangle \qquad m_2 = \frac{-6}{3} = -2$$

$$(m_1)(m_2) = \left(\frac{1}{2}\right)(-2) = -1$$

slopes are opposite reciprocal
hence lines are $\perp^r$ to each other

At point of intersection, lines share a point (x, y)

$$2\times(-4+2t = 2+3s)$$
$$5+t = 4-6s$$

$$-8+4t = 4+6s$$
$$5+t = 4-6s$$
$$-3+5t = 8+0$$
$$5t = 11$$
$$t = 11/5$$

$$\bar{r}_1\left(t = \frac{11}{5}\right) = \left\langle \frac{2}{5}, \frac{36}{5} \right\rangle$$

$\boxed{\text{pt. of intersection: } \left(\frac{2}{5}, \frac{36}{5}\right)}$ Ans.