Week in Review 7

Monday, October 17, 2022 4:53 PM

Q1. motion : $s = t^3 - 8t^2 + 24t$, $t \geq 0$.

(a) $v(t) = s'(t) = 3t^2 - 16t + 24$

(b) $v(1) = 3 \cdot 1^2 - 16 \cdot 1 + 24 = 11$ ft/s

(c) at rest : $v = 0$

Set up $v(t) = 3t^2 - 16t + 24 = 0$

Solve for t: $(-16)^2 - 4 \times 3 \times 24 = -32 < 0$.

We don't have a solution

So the particle never rest. (velocity is never 0).

(d) positive direction : $s \geq 0$

Set up $s(t) = t^3 - 8t^2 + 24t \geq 0$

Solve for t: $t(\underbrace{t^2 - 8t + 24}_{\text{we cannot factorize further}}) \geq 0$.

$\Rightarrow \quad t \geq 0$

It's always positive. The particle always moves in positive direction.

(e). Total distance traveled in first 6 s.

Based (d), there is no cancellation for position function.

$\Rightarrow$ distance = difference of position

$= s(6) - s(0)$

$= 6^3 - 8 \cdot 6^2 + 24 \cdot 6 - 0 = 72$ ft.

(f). Find acceleration. $a(t)$. and $a(1)$.

$a(t) = v'(t) = 6t - 16$.

$a(t) = v'(t) = 6t - 16$.

$a(1) = 6 - 16 = -10$. ft/s^2.

(g) speeding up: $a(t) > 0$

set $6t - 16 > 0$

solve for $t \Rightarrow t > \frac{16}{6} \Rightarrow t > \frac{8}{3}$.

When $t > \frac{8}{3}$, it is speeding up.

slowing down: $a(t) < 0$.

set $6t - 16 < 0$.

$\Rightarrow t < \frac{8}{3}$. $(t \geq 0) \Rightarrow 0 \leq t < \frac{8}{3}$.

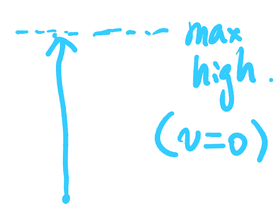
When $0 \leq t < \frac{8}{3}$, it is slowing down.

Q2. $h = 2 + 24.5t - 4.9t^2$.

(a). $v(t) = h'(t) = 24.5 - 9.8t$

$v(2) = 24.5 - 9.8 \times 2 = 4.9$ m/s

and $v(4) = 24.5 - 9.8 \times 4 = -14.7$ m/s.



(b) Set up $v(t) = 0$

$\Rightarrow 24.5 - 9.8t = 0$.

solve for t: $t = \frac{24.5}{9.8} = 2.5$ s.

It reaches max height at $t = 2.5$ s.

(c) $h(2.5) = 2 + 24.5 \times 2.5 - 4.9 \times (2.5)^2 = 32.625$ m.

max height at 32.625 m.

$h(t)=0$.

(d) hit the ground?

Set up $h(t) = 2 + 24.5t - 4.9t^2 = 0$

Solve for t: $t = \dfrac{-24.5 \pm \sqrt{(24.5)^2 - 4 \times 2 \times (-4.9)}}{2 \cdot (-4.9)}$

≈ 5.08 s (rule out negative solution).

About 5.08 s, it hits the ground.

(e) $v(5.08) = 24.5 - 9.8 \times 5.08 \approx -25.3$ m/s.

Q3.

$A(r) = \pi r^2$

(a) average rate of change:

① $\dfrac{A(3) - A(2)}{3-2} = \dfrac{\pi \cdot 3^2 - \pi \cdot 2^2}{3-2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$.

② $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{(2.5)^2\pi - 2^2\pi}{2.5-2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

③ $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{\pi(2.1)^2 - \pi \cdot 2^2}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) instantaneous rate of change:

(b) instantaneous rate of change:

$$A'\big|_{r=2} = A'(2) = 2\pi r\big|_{r=2} = 4\pi.$$

Q4: surface area : 

$S = 4\pi r^2$

instant rate of increase area: $\frac{dS}{dr} = 8\pi r$

(a) $r=1$: $\frac{dS}{dr}\big|_{r=1} = 8\pi$ ft^2/ft

(b) $r=2$: $\frac{dS}{dr}\big|_{r=2} = 16\pi$ ft^2/ft

(c) $r=3$: $\frac{dS}{dr}\big|_{r=3} = 24\pi$ ft^2/ft.

Q5. It has constant relative growth rate k

$\frac{dP}{dt} = k \cdot P$ $\qquad$ P : population of bacteria.

$$\Rightarrow \quad P(t) = P(0)\,e^{kt}$$

Find $P(0)$ and k :

$$\begin{cases} P(1) = 1000 \\ P(5) = 3500 \end{cases} \Rightarrow \begin{cases} P(0)\,e^{k} = 1000 & ① \\ P(0)\,e^{5k} = 3500 & ② \end{cases}$$

②/① : $e^{4k} = \frac{3500}{1000} = 3.5 \Rightarrow 4k = \ln(3.5)$

$$\Rightarrow k = \tfrac{1}{4}\ln(3.5).$$

Use ① : $P(0)\,e^{\frac{1}{4}\ln(3.5)} = 1000 \Rightarrow P(0) = \frac{1000}{e^{\frac{1}{4}\ln(3.5)}} \approx 731.1$

Use ①: $P(0)\, e^{\frac{1}{4}\ln(3.5)} = 1000 \Rightarrow P(0) = \frac{\quad}{e^{\frac{1}{4}\ln(3.5)}} \approx 731.1$

So formula is $\boxed{P(t) = 731.1\, e^{\frac{t}{4}\ln 3.5}}$

And $P(2) = 731.1\, e^{\frac{2}{4}\ln 3.5} \approx 1367.8$

$$\left.\frac{dP}{dt}\right|_{t=2} = k \cdot P(2) \approx \frac{1}{4}\ln(3.5) \times 1367.8 \approx 428.4$$

Q6: $A = A(t)$ is amount remaining.

$$\frac{dA}{dt} = kA \Rightarrow A(t) = A_0 e^{kt}.$$

$$A_0 = 500 \qquad t_{half} = 15$$

Half-life formula: $t_{half} = \frac{\ln \frac{1}{2}}{k}$.

$$\Rightarrow k = \frac{\ln \frac{1}{2}}{t_{half}} = \frac{1}{15}\ln\frac{1}{2}.$$

So $A(t) = 500\, e^{\frac{t}{15}\ln\frac{1}{2}}$

Set up $500\, e^{\frac{t}{15}\ln\frac{1}{2}} = 125$. solve for t

$$\Rightarrow e^{\frac{t}{15}\ln\frac{1}{2}} = \frac{125}{500} = \frac{1}{4}.$$

$$\Rightarrow \frac{t}{15}\ln\frac{1}{2} = \ln\frac{1}{4}.$$

$$\Rightarrow \frac{t}{15} = \frac{\ln\frac{1}{4}}{\ln\frac{1}{2}} = \frac{2\ln\frac{1}{2}}{\ln\frac{1}{2}} = 2$$

$$t = 15 \cdot 2 = 30 \text{ hr}.$$

Q7. Cooling model: $\frac{dT}{dt} = k(T - T_r) \Rightarrow T(t) = T_r + (T_0 - T_r)e^{kt}$

Q7. Cooling model: $\frac{dT}{dt} = k(T - T_r) \Rightarrow T(t) = T_r + (T_0 - T_r)e^{\cdots}$

$T_0 = 375 \quad T_r = 75.$ So $T(t) = 75 + 300e^{kt}.$

$T(30) = 175 \Rightarrow 75 + 300e^{30k} = 175$

So $300e^{30k} = 100$

$\Rightarrow e^{30k} = \frac{1}{3} \Rightarrow 30k = \ln\frac{1}{3}$

So $k = \frac{1}{30}\ln\frac{1}{3}.$

Then $T(t) = 75 + 300e^{\frac{t}{30}\ln\frac{1}{3}}$

Set up $75 + 300e^{\frac{t}{30}\ln\frac{1}{3}} = 90$

solve for t: $300e^{\frac{t}{30}\ln\frac{1}{3}} = 15$

$\Rightarrow e^{\frac{t}{30}\ln\frac{1}{3}} = \frac{15}{300} = \frac{1}{20}.$

$\frac{t}{30}\ln\frac{1}{3} = \ln\frac{1}{20} \Rightarrow t = \frac{30\ln\frac{1}{20}}{\ln\frac{1}{3}} \approx 81.8$ min

---

Q8. Warming model: $\frac{dT}{dt} = k(T - T_r)$

$\Rightarrow T(t) = T_r + (T_0 - T_r)e^{kt}$

$T_0 = 5, \quad T_r = 20 \Rightarrow T(t) = 20 - 15e^{kt}.$

$T(25) = 10 \Rightarrow 20 - 15e^{25k} = 10.$

So $-15e^{25k} = -10 \Rightarrow e^{25k} = \frac{-10}{-15} = \frac{2}{3}$

$25k = \ln\frac{2}{3}$ So $k = \frac{1}{25}\ln\frac{2}{3}.$

Then $T(t) = 20 - 15e^{\frac{t}{25}\ln\frac{2}{3}}$

Then $T(t) = 20 - 15e^{\frac{t}{25}\ln\frac{2}{3}}$

(a) $T(50) = 20 - 15e^{\frac{50}{25}\ln\frac{2}{3}} = 20 - 15e^{\ln\left(\frac{2}{3}\right)^2}$

$= 20 - 15 \cdot \frac{4}{9} = \frac{40}{3}$ °C

(b) Set up $T(t) = 15$

So $20 - 15e^{\frac{t}{25}\ln\frac{2}{3}} = 15$

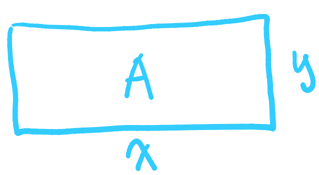
Solve for t: $-15e^{\frac{t}{25}\ln\frac{2}{3}} = -5$

$\Rightarrow e^{\frac{t}{25}\ln\frac{2}{3}} = \frac{-5}{-15} = \frac{1}{3}$

$\frac{t}{25}\ln\frac{2}{3} = \ln\frac{1}{3} \quad \Rightarrow \quad t = \frac{25\ln\frac{1}{3}}{\ln\frac{2}{3}} \approx 67.74$ min

Related Rates:

Q 9.



So $A = xy$.

$x = x(t)$, $y = y(t)$, $A = A(t)$.

Set up $A = xy$.

differentiate it with respect to time t:

$$A' = x'y + xy'$$

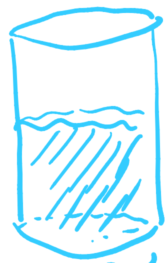
Find: $A'\Big|_{\substack{x=20\\ y=10}} = x' \cdot 10 + 20 \cdot y'$ $\quad \begin{cases} x' = 8 \text{ cm/s} \\ y' = 3 \text{ cm/s} \end{cases}$

$= 8 \cdot 10 + 20 \cdot 3$

$= 8 \cdot 10 + 20 \cdot 3$

$= 80 + 60 = 140 \quad cm^2/s.$

Q 10.



$V = V(t) = \pi r^2 \cdot h$

$= \pi \cdot 5^2 \cdot h$

$= 25 \pi h.$

$V' = 3 \ m^3/min, \quad h' = ?$

Set up $V = 25\pi h$

Differentiate with respect to time t:

$$V' = 25\pi h'$$

Solve for h': $\quad h' = \frac{V'}{25\pi} = \frac{3}{25\pi} \quad m/min$

Q 11.

S : surface area.

$S = 4\pi r^2. \qquad \left.\frac{dS}{dt}\right|_{r=8} = ?$

Set up $S = 4\pi r^2.$

Differentiate with respect to time t:

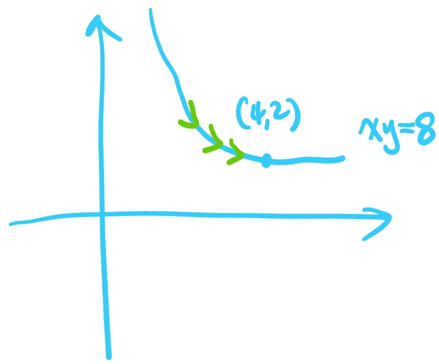
$$S' = 4\pi \cdot 2r \cdot r'$$

We know $r' = 2$ cm/min

$$S'|_{r=8} = 4\pi \cdot 2 \cdot 8 \cdot 2 = 128\pi \quad cm^2/min$$

$$S'\big|_{r=8} = 4\pi \cdot 2 \cdot 8 \cdot 2 = 128\pi \ \text{cm}^2/\text{min}$$

Q12.



$y' = -3$ cm/s (decreasing)

$$x'\big|_{x=4,\, y=2} = ?$$

Set up $xy = 8$.

Differentiate with respect to time t:

$$x'y + xy' = 0$$

Solve for x': $x' = \dfrac{-xy'}{y}$

Plug in: $$x'\big|_{x=4,\, y=2} = -\frac{4 \cdot y'}{2} = -\frac{4 \cdot (-3)}{2} = 6 \ \text{cm/s}.$$

Q13. Triangle:

area A

Set up $A = \frac{1}{2} \cdot b \cdot h$

Differentiate with respect to time t:

$$A' = \frac{1}{2} b' \cdot h + \frac{1}{2} b \cdot h'$$

$$A' = \frac{1}{2} b' \cdot h + \frac{1}{2} b \cdot h'$$

Goal: $\quad b' \Big|_{\substack{h=10 \\ A=100}} = ?$

Solve for b': $\quad b' = \dfrac{2A' - b \cdot h'}{h}$

Plug in $\quad b' \Big|_{\substack{h=10 \\ A=100}} = \dfrac{2A' - b \cdot h'}{10}$

$\begin{cases} h' = 1 \ \text{cm/min} \\ A' = 2 \ \text{cm}^2/\text{min} \end{cases}$

$$= \frac{2 \cdot 2 - b \cdot 1}{10}$$

use $A = \frac{1}{2} bh$ for b when $\begin{cases} A = 100 \\ h = 10 \end{cases}$.

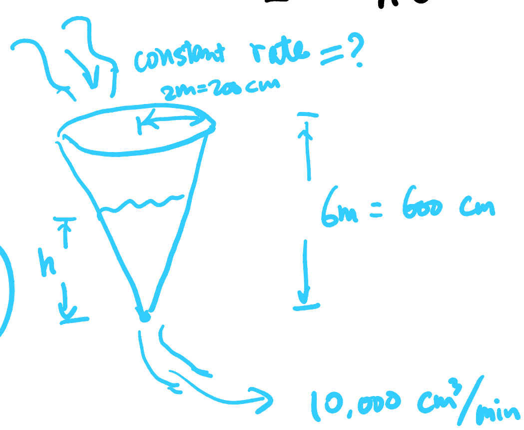
So $b = \frac{2A}{h} = \frac{200}{10} = 20$

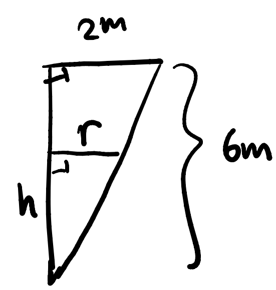
$$= \frac{4 - 20 \cdot 1}{10}$$

$$= -1.6 \ \text{cm/min}$$

Q 14.

$h' \Big|_{h=200} = 20$ cm/min





Set up $\quad V = \frac{1}{3} \pi r^2 \cdot h$

By similar triangle: $\quad \frac{r}{h} = \frac{2}{6} = \frac{1}{3}$

$$\Rightarrow \quad r = \frac{1}{3} h.$$

$$\Rightarrow \quad r = \frac{1}{3}h.$$

So $\quad V = \frac{1}{3} \cdot \pi \cdot \left(\frac{1}{3}h\right)^2 \cdot h = \frac{1}{27}\pi h^3.$

Differentiate with respect to time t:

$$V' = \frac{1}{27}\pi \cdot 3 \cdot h^2 \cdot h' = \frac{\pi}{9} h^2 \cdot h'$$

When $h = 200$, $\quad h'\big|_{h=200} = 20$

$$\Rightarrow \quad V'\big|_{h=200} = \frac{\pi}{9} \cdot (200)^2 \cdot 20 = \frac{800{,}000\pi}{9}$$

pumped rate of water: $\quad \frac{800{,}000\pi}{9} + 10{,}000 \approx 2.89 \times 10^5 \ cm^3/min$