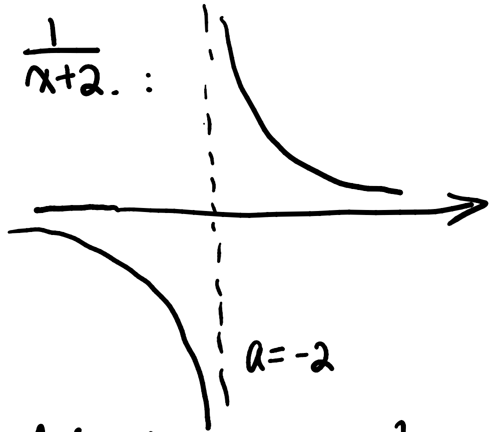
Week in Review 3

Monday, September 19, 2022 4:58 PM

Q1. (a) graph of $\frac{1}{x+2}$ :




$x=-2$ is NOT defined, so $\frac{1}{x+2}$ is not continuous at $-2$.

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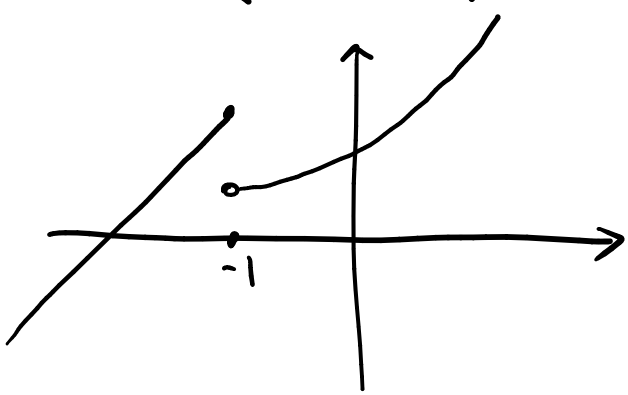
(b) $f(x) = \begin{cases} x+3 & \text{if } x \le -1 \\ 2^x & \text{if } x > -1 \end{cases}$.




Because $\lim_{x\to -1^+} f(x) = 2^{-1} = \frac{1}{2} \neq \lim_{x \to -1^-} f(x) = (-1)+3 = 2$. (jump).

So $f$ is NOT continuous at $-1$.

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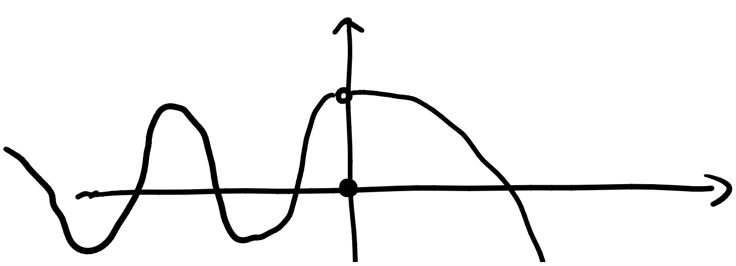
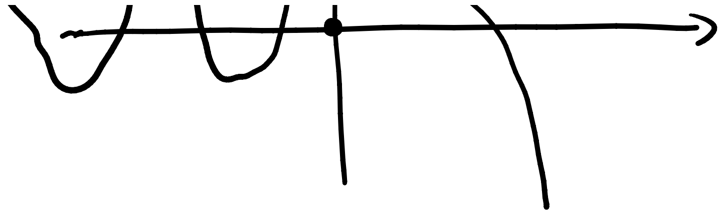
(c) $f(x) = \begin{cases} \cos x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1-x^2 & \text{if } x>0 \end{cases}$



$\lim_{x\to 0^-} f(x) = \cos(0) = 1$

$= \lim_{x\to 0^+} f(x) = 1-0^2 = 1$



$= \lim_{x\to 0^+} f(x) = 1 - 0^2 = 1$

but $f(0) = 0 \neq 1$.

So it is NOT continuous at 0.

( It's a removable discontinuity at 0 ).

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Q 2. (a) Compute $\lim_{x\to 4} 3^{\sqrt{x^2-2x-4}}$

$$= 3^{\left(\lim_{x\to 4}\sqrt{x^2-2x-4}\right)}.$$

$$= 3^{\sqrt{\lim_{x\to 4}(x^2-2x-4)}}$$

$$= 3^{\sqrt{4^2-2\cdot 4-4}}$$

$$= 3^2 = \boxed{9}$$

$3^x$ is continuous. So we can bring the limit inside

$\sqrt{x}$ is continuous bring the limit inside.

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(b) Compute $\lim_{x\to\pi} \sin(x+\sin x)$.

$$= \sin\left(\lim_{x\to\pi}(x+\sin x)\right)$$

$$= \sin(\pi + \sin\pi).$$

$$= \sin(\pi + 0) = \boxed{0}$$

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Q 3. Find a and b so that the function

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$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x<1 \\ ax^2+bx-5 & \text{if } 1\le x<2 \\ 3x-a+2b & \text{if } x\ge 2. \end{cases}$$

is continuous everywhere.

Solution. For $x<1$, $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$

$f$ is continuous for all $x<1$.

For $1<x<2$, $ax^2+bx-5$ polynomial

$f$ is continuous for all $1<x<2$.

For $2<x$, $3x-a+2b$ linear function

$f$ is continuous for all $x>2$.

Only problematic points are $x=1$ and $x=2$.

① For $x=1$ :

$$\left.\begin{aligned} &\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+1) = 1+1=2. \\ &\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax^2+bx-5 = a+b-5 = f(1). \end{aligned}\right\} \text{same}$$

So $f$ is continuous at $x=1$ if $2 = a+b-5$

② For $x=2$ :

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2+bx-5 = 4a+2b-5 \quad \Big\} \text{same}$$

② For $x=2$ :

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2+bx-5 = 4a+2b-5$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x-a+2b = 6-a+2b = f(2).$$
} same

So $f$ is continuous at $x=2$ if $4a+2b-5 = 6-a+2b$

$$\begin{cases} a+b-5=2 \\ 4a+2b-5 = 6-a+2b \end{cases} \rightsquigarrow 4a-5=6-a \Rightarrow 5a=11$$

$$\Rightarrow \boxed{a = \frac{11}{5}}$$

plug back in first equation : $\frac{11}{5}+b-5=2$.

$$\Rightarrow \boxed{b = 7-\frac{11}{5} = \frac{24}{5}}$$

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Q4 : Show that the equation $e^x = 3-2x$ has a root in the interval $(0, 1)$.

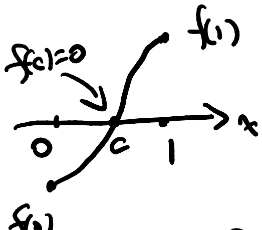
<u>Solution</u> / Proof : Rewrite the equation $e^x+2x-3=0$.

Set $f(x) = e^x+2x-3$ (whole expression on the left).

Notice that $f(x) = e^x+2x-3$ is continuous on $[0,1]$ (closed interval).

Compute/Evaluate ⟨1⟩ $f(0) = e^0+2\cdot 0-3 = 1-3 = -2 < 0$

<2> $f(1) = e^1 + 2\cdot 1 - 3 = e - 1 > 0$




$f(0) < 0 < f(1)$

left end    right end.

By intermediate value theorem, there is a $c$ in $(0,1)$.

So that $f(c) = 0$

$\Rightarrow \quad e^c + 2c - 3 = 0$.

So $e^x + 2x - 3 = 0$ has a root $x = c$.

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Q5 Compute limits :

(a) $\lim\limits_{x\to\infty} \dfrac{x+1}{4x-3}$    ( divide leading power of denominator ).

$= \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}}{4-\frac{3}{x}}$

$= \dfrac{1+0}{4-0} = \boxed{\frac{1}{4}}$

(b) $\lim\limits_{x\to-\infty} \dfrac{x^2-1}{2x+5}$

$= \lim\limits_{x\to-\infty} \dfrac{x-\frac{1}{x}}{2+\frac{5}{x}}$

$$= \frac{-\infty - 0}{2+0} = \boxed{-\infty}$$

(c) $$\lim_{x \to -\infty} \frac{\sqrt{2x^2+3x+1}}{x-1}$$

$$= \lim_{x \to -\infty} \frac{\frac{\sqrt{2x^2+3x+1}}{x}}{1-\frac{1}{x}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{2+\frac{3}{x}+\frac{1}{x^2}}}{1-\frac{1}{x}}$$

$$= \frac{-\sqrt{2+0+0}}{1-0} = \boxed{-\sqrt{2}}$$

$x=-5$

$$\frac{\sqrt{2}}{-5} = -\sqrt{\frac{2}{(-5)^2}}$$

$(x<0)$

$$-\sqrt{\frac{2x^2+3x+1}{x^2}}$$

$$= -\sqrt{2+\frac{3}{x}+\frac{1}{x^2}}$$

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(d) $$\lim_{x \to \infty} \frac{1+e^x}{1-3e^x}$$

("leading power" of denominator is "$e^x$".)

$$= \lim_{x \to \infty} \frac{\frac{1}{e^x}+1}{\frac{1}{e^x}-3}$$

$$= \frac{0+1}{0-3} = \boxed{-\frac{1}{3}}$$

(e) $$\lim_{x \to \infty} \left[\ln(3x^2+4) - \ln(6x^2-5)\right]$$

property:

$\ln A - \ln B$

$= \ln \frac{A}{B}$

$$= \lim \ln\left(\frac{3x^2+4}{\right.$$

$= \ln\frac{A}{B}$

$$= \lim_{x\to\infty} \ln\left(\frac{3x^2+4}{6x^2-5}\right)$$

$$= \ln\left(\lim_{x\to\infty} \frac{3x^2+4}{6x^2-5}\right)$$

$$= \ln\left(\lim_{x\to\infty} \frac{3+\frac{4}{x^2}}{6-\frac{5}{x^2}}\right).$$

$$= \ln\left(\frac{3+0}{6-0}\right) = \boxed{\ln\frac{1}{2}}$$

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(f). $$\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$$

$(a-b)(a+b) = a^2-b^2$

$a = \sqrt{x^2+x+1}$
$b = x$

$$= \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+x+1}+x\right)\left(\sqrt{x^2+x+1}-x\right)}{\sqrt{x^2+x+1}-x}$$

$$= \lim_{x\to-\infty} \frac{(\cancel{x^2}+x+1)-\cancel{x^2}}{\sqrt{x^2+x+1}-x}$$

$$= \lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+x+1}-x}$$

$$= \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{\frac{\sqrt{x^2+x+1}}{x}-1}$$

$$= \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{-\sqrt{x^2+x+1} \ldots}$$

$$= \lim_{x \to -\infty} \frac{1 + x}{-\sqrt{\frac{x^2+x+1}{x^2}} - 1}$$

$$= \lim_{x \to -\infty} \frac{1 + \frac{1}{x}}{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1} = \frac{1+0}{-\sqrt{1} - 1} = \boxed{-\frac{1}{2}}$$

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Q 6. Find horizontal and vertical asymptotes of

$$f(x) = \frac{2e^{x}}{e^x - 5}.$$

Horizontal : look at $\lim_{x\to\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$$\lim_{x\to\infty} \frac{2e^x}{e^x-5} = \lim_{x\to\infty} \frac{2}{1-\frac{5}{e^x}} = \frac{2}{1-0} = 2 \quad \boxed{\text{finite number}}$$

$$\lim_{x\to-\infty} \frac{2e^x}{e^x-5} = \frac{2\cdot 0}{0-5} = \frac{0}{-5} = 0 \quad \boxed{\text{finite number}}$$

$y=0$ and $y=2$ are horizontal asymptotes.

Vertical : candidate(s) are zeros of denominator

set $e^x - 5 = 0$. then solve for $x$.

$\Rightarrow \quad e^x = 5 \quad \Rightarrow \quad x = \ln 5$

examine/verify : $\lim_{x\to \ln 5^+} f(x) = \lim_{x\to\ln 5^+} \frac{2e^x}{e^x-5}$

examine/verify : $\lim_{x \to \ln 5^+} f(x) = \lim_{x \to \ln 5^+} \frac{2e^x}{e^x - 5}$

$$= \frac{2 \cdot e^{\ln 5}}{e^{\ln 5^+} - 5}$$

$2 \cdot 5 = 10$

$0^+$

$$= \infty \qquad (\text{vertical asymptote})$$

So $x = \ln 5$ is a vertical asymptote.

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Q7. Find the equation of tangent line to graph of $f(x) = \sqrt{x}$ at $(1, f(1))$.

Solution. Find the slope $m = f'(1)$

So $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ (by definition).

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)}.$$

$$\boxed{(a-b)(a+b) = a^2 - b^2}$$

$$= \lim_{h \to 0} \frac{(\cancel{1}+h) - \cancel{1}}{h(\sqrt{1+h} + 1)}.$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h} + 1)}.$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}.$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

point is $(1, f(1)) = (1, 1)$.

Use point-slope formula : $y - y_0 = m(x - x_0)$.

$$y - 1 = \frac{1}{2}(x-1)$$

$$\Rightarrow \boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

$y - 1 = \frac{1}{2}x - \frac{1}{2}$

$y = \frac{1}{2}x - \frac{1}{2} + 1$

$= \frac{1}{2}x + \frac{1}{2}$.

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Q8. Find equation of tangent line of $y = g(x)$ at $x = 5$.

information : $g(5) = -3$ and $g'(5) = 4$.

<u>Solution</u>. slope : $m = g'(5) = 4$.

point : $(5, g(5)) = (5, -3)$

Use point-slope formula : $y - y_0 = m(x - x_0)$

$$y - (-3) = 4(x - 5)$$

$$\Rightarrow \boxed{y = 4x - 23}$$

$y + 3 = 4x - 20$

$y = 4x - 23$

Q 9. Position function $s(t) = 2t^2 - 6t + 5$.

(a) find average speed over $[4,6]$

(b) instant velocity at $t=4$.

(a) $$\text{average} = \frac{s(6) - s(4)}{6-4}$$

$$= \frac{(2\cdot6^2 - 6\cdot6+5) - (2\cdot4^2 - 6\cdot4+5)}{2}$$

$$= \frac{(72-36+5) - (32-24+5)}{2}$$

$$= \frac{41-13}{2} = \frac{28}{2} = \boxed{14}$$

(b)

$$\text{instant velocity} = v(4) = s'(4).$$

$$= \lim_{h\to0} \frac{s(4+h) - s(4)}{h} \quad \text{(definition)}$$

$$= \lim_{h\to0} \frac{[2(4+h)^2 - 6(4+h)+5] - 13}{h}$$

in (a) we computed $S(4)=13$

$$= \lim_{h\to0} \frac{[2(16+8h+h^2) - 24 - 6h + 5] - 13}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{32} + 16h + 2h^2 - \cancel{24} - 6h + \cancel{5} - \cancel{13}}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 10h}{h}$$

$$= \lim_{h\to 0} (2h+10) = 2\cdot 0 + 10 = \boxed{10}$$

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Q 10 : (a) $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6}+h\right) - \frac{1}{2}}{h}$ find f & a.

Write out definition : $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$.

$\Rightarrow$ $f(a+h) = \sin\left(\frac{\pi}{6}+h\right)$.

$\Rightarrow$ $f(x) = \sin(x)$ and $a = \frac{\pi}{6}$.

( check : $f(a) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ ) ✓

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(b) $\lim_{x\to \frac{1}{4}} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$. find f and a.

write out definition ( different form) $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$.

$\Rightarrow \quad a = \frac{1}{4}. \qquad f(x) = \frac{1}{x}.$

( check : $f(a) = \frac{1}{\frac{1}{4}} = 4 \quad \checkmark$ ).