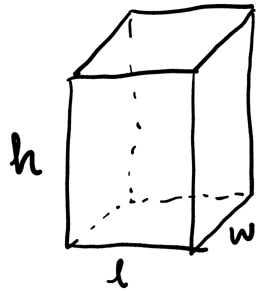
Week in Review 11

Monday, November 14, 2022 5:01 PM

Q 1.



(diagram).

$V = 10$

$l = 2w$

$V = l w \cdot h = 10$

$= 2w^2 h$

Cost :

$$C = \underbrace{l \cdot w \cdot 10}_{\text{base}} + \underbrace{h \cdot l \times 2 \times 6}_{\text{front + back}} + \underbrace{h w \times 2 \times 6}_{\text{left + right}}$$

$$= 20w^2 + 12 l \cdot h + 12 h w \qquad (l = 2w).$$

$$= 20w^2 + 24 hw + 12 hw$$

$$= 20w^2 + 36 hw$$

(Volume : $2w^2 h = 10 \Rightarrow h = \frac{5}{w^2}$).

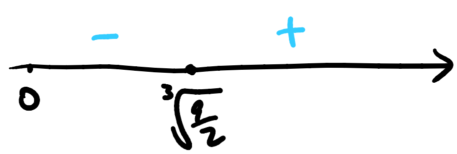
$$C(w) = 20w^2 + 36 \cdot w \cdot \frac{5}{w^2} = 20w^2 + \frac{180}{w} \qquad (0 < w < \infty)$$

Minimize it : $\boxed{C'(w) = 40w - \frac{180}{w^2}}$, set $C'(w) = 0$.

$$\Rightarrow 40w = \frac{180}{w^2} \Rightarrow w^3 = \frac{180}{40} = \frac{9}{2}$$

$$\Rightarrow w = \sqrt[3]{\frac{9}{2}} \quad \text{(candidate)}.$$

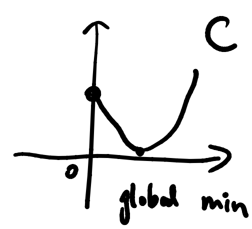
sign of C' :



At $w = \sqrt[3]{\frac{9}{2}}$, C has a local min.

This is the only local min,

since C is first decreasing, then increasing



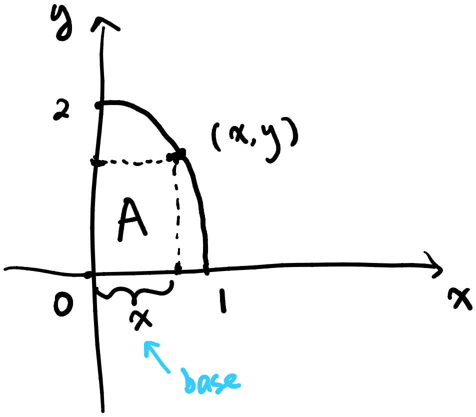
Since C is first decreasing, then increasing.

It must be the global minimum.

Dimensions: width $\sqrt[3]{\frac{9}{2}}$, length $2\cdot\sqrt[3]{\frac{9}{2}}$. height: $\frac{5}{3}\sqrt[3]{\frac{4}{3}}$.

(plug $w=\sqrt[3]{\frac{9}{2}}$ back to $h=\frac{5}{w^2}$)

Q2.

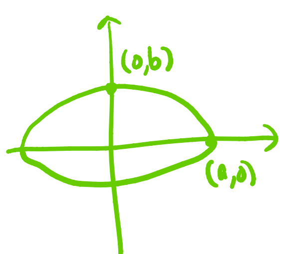


(diagram)

$A = xy$ We want to maximize it.

(x,y) satisfies: $x^2 + \frac{y^2}{4} = 1$ (this point is on the curve).

Standard ellipse eq: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Solve for y from the ellipse:

$\Rightarrow\ 4x^2 + y^2 = 4. \Rightarrow\ y^2 = 4 - 4x^2$

$\Rightarrow\ y = \sqrt{4-4x^2}$ (1^{st} quadrant, $y>0$).

So $A = A(x) = x\cdot\sqrt{4-4x^2}$ $(0 \le x \le 1)$

Compute: $A'(x) = \sqrt{4-4x^2} + x\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{4-4x^2}}\cdot(-8x)$

$= \sqrt{4-4x^2} + \frac{-4x^2}{\ldots}$

$$= \sqrt{4-4x^2} + \frac{-4x^2}{\sqrt{4-4x^2}}$$

$$= \frac{(4-4x^2) - 4x^2}{\sqrt{4-4x^2}} = \frac{4-8x^2}{\sqrt{4-4x^2}}.$$

Set $A'(x) = 0 \Rightarrow 4-8x^2 = 0 \Rightarrow x^2 = \frac{4}{8} = \frac{1}{2}$

$\Rightarrow x = \sqrt{\frac{1}{2}} \quad (0 \le x \le 1)$.

1st method: Compare $A(0) = 0$, $A(1) = 0$, $A(\sqrt{\frac{1}{2}}) > 0$

A has global max at $\boxed{x = \sqrt{\frac{1}{2}}}$

2nd method: sign of A' :
$$\begin{array}{ccccc} & + & & - & \\ 0 & & \sqrt{\tfrac{1}{2}} & & 1 \end{array}$$
only local max.

$\Rightarrow$ A has a global max at $x = \sqrt{\frac{1}{2}}$

Q3. (a) $f(x) = \frac{5}{\sqrt{1-x^2}} - \frac{7+3x-x^4}{x} + \frac{1}{1+x^2}$.

$$= 5 \cdot \frac{1}{\sqrt{1-x^2}} - \frac{7}{x} - 3 + x^3 + \frac{1}{1+x^2}.$$

So anti-derivative: $F(x) = 5 \cdot \arcsin(x) - 7\ln|x| - 3x + \frac{1}{4}x^4 + \arctan(x) + C$

(b) $f(x) = 3x^2(x^3+1)$

$$= 3x^5 + 3x^2$$

$$= 3x^5 + 3x^2$$

anti-derivative : $F(x) = \frac{1}{2}x^6 + x^3 + C$

(c) $f(x) = \dfrac{2x^2+6}{x^3}$

$$= \frac{2}{x} + \frac{6}{x^3}$$

anti-derivative : $F(x) = 2 \cdot \ln|x| - 3x^{-2} + C$

(d) $f(x) = \csc(x)\,(\cot(x) - \csc(x))$

$$= \csc(x)\cot(x) - \csc^2(x)$$

anti-derivative : $F(x) = -\csc(x) + \cot(x) + C$

(e) $f(x) = 7^x + \dfrac{1}{5x^3} + \sqrt[5]{x^3}$

$$= 7^x + \frac{1}{5}\cdot x^{-3} + x^{\frac{3}{5}}$$

anti-derivative : $F(x) = \dfrac{7^x}{\ln 7} - \dfrac{1}{10}x^{-2} + \dfrac{5}{8}x^{\frac{8}{5}} + C$

$$\left(\frac{5}{8}\cdot x^{\frac{8}{5}}\right)' = \frac{5}{8}\cdot\frac{8}{5}\cdot x^{\frac{3}{5}}$$

$$= x^{\frac{3}{5}}$$

$$x^{\alpha} \xrightarrow{\text{anti-deri.}} \frac{x^{\alpha+1}}{\alpha+1}$$

$x^{\alpha} \xrightarrow{\text{anti-deriv.}} \dfrac{x^{\alpha+1}}{\alpha+1}$

Q4. (a) $f'(x) = 2(1-x^2)^{-1/2} + e^x$ with $f(0) = 4$.

$$f'(x) = 2 \cdot \frac{1}{\sqrt{1-x^2}} + e^x \Rightarrow f(x) = 2 \cdot \arcsin(x) + e^x + C$$

plug in $x=0$: $f(0) = 2 \cdot \arcsin(0) + e^0 + C$

$$= 0 + 1 + C$$

$$\Rightarrow \quad 1 + C = f(0) = 4 \quad \Rightarrow \quad C = 3$$

Therefore, $\boxed{f(x) = 2\arcsin(x) + e^x + 3}$

(b). $f'(x) = 2e^x - 5$ with $f(0) = 1$.

$$f(x) = 2e^x - 5x + C$$

plug in $x=0$: $f(0) = 2e^0 - 5 \cdot 0 + C$

$$= 2 + C$$

So $\quad 2 + C = f(0) = 1 \quad \Rightarrow \quad C = -1.$

Therefore, $\boxed{f(x) = 2e^x - 5x - 1}$

(c) $f''(x) = 5x^4 - 6$ with $f'(0) = 4$ and $f(1) = 2$.

$$f'(x) = x^5 - 6x + C_1 \qquad (f'(0) = 4)$$

plug in $x=0$: $f'(0) = 0^5 - 6 \cdot 0 + C_1 = C_1$

plug in $x=0$: $\quad f'(0) = 0^5 - 6\cdot 0 + C_1 = C_1$, $\quad f'(0)=4$ $\Big\} \Rightarrow C_1 = 4.$

So $f'(x) = x^5 - 6x + 4$

$$\Rightarrow \quad f(x) = \tfrac{1}{6}x^6 - 3x^2 + 4x + C_2 \qquad (f(1)=2)$$

plug in $x=1$: $\quad f(1) = \tfrac{1}{6}\cdot 1^6 - 3\cdot 1^2 + 4\cdot 1 + C_2 = \tfrac{7}{6} + C_2$, $\quad f(1) = 2$ $\Big\} \Rightarrow C_2 = \tfrac{5}{6}.$

Therefore, $\boxed{f(x) = \tfrac{1}{6}x^6 - 3x^2 + 4x + \tfrac{5}{6}}$

(d) $\quad f''(x) = 20x^3 + 6e^x \quad$ with $f(0)=4$ and $f(1)=2$

So $\quad f'(x) = 5x^4 + 6e^x + C_1$

Then $\quad f(x) = x^5 + 6e^x + C_1 x + C_2$

So $\quad f(0) = 0^5 + 6\cdot e^0 + C_1\cdot 0 + C_2$

$$= 6 + C_2$$

$$f(1) = 1^5 + 6\cdot e^1 + C_1\cdot 1 + C_2$$

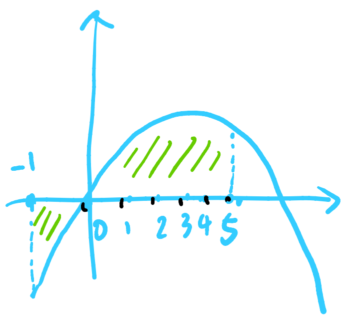
$$= 6e + (1 + C_1) + C_2$$

$f(0) = 4$ and $f(1) = 2.$

$$\Rightarrow \begin{cases} 6 + C_2 = 4 \\ 6e + 1 + C_1 + C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = 3 - 6e \\ C_2 = -2 \end{cases}$$

Therefore, $\boxed{f(x) = x^5 + 6e^x + (3-6e)x - 2.}$

Q5 $\quad f(x) = 7x - x^2 \quad$ on $[-1, 5]$.



$$\Delta x = \frac{b-a}{n} = \frac{5-(-1)}{6} = 1$$

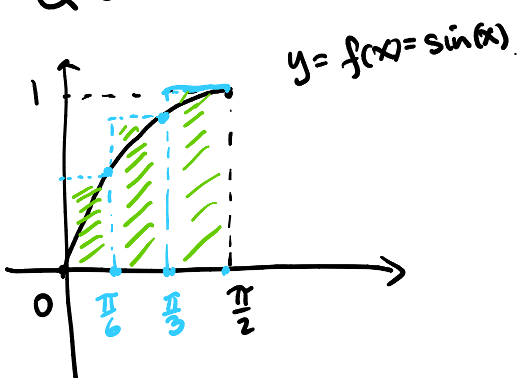
right endpoints: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, $x_5 = 4$. $x_6 = 5$.

$$A \approx R_6 = [f(0) + f(1) + f(2) + f(3) + f(4) + f(5)]\Delta x$$

$$= (0 + 6 + 10 + 12 + 12 + 10) \cdot 1$$

$$= \boxed{50}$$

Q6.



$$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6}.$$

right-endpoints: $x_1 = \frac{\pi}{6}$, $x_2 = \frac{\pi}{3}$, $x_3 = \frac{\pi}{2}$.

$$A \approx R_3 = [f(x_1) + f(x_2) + f(x_3)]\Delta x$$

(It's over estimate.).

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$$A \approx R_3 = \left[f(x_1) + f(x_2) + f(x_3)\right]\Delta x$$

$$= \left[\sin\left(\tfrac{\pi}{6}\right) + \sin\left(\tfrac{\pi}{3}\right) + \sin\left(\tfrac{\pi}{2}\right)\right] \cdot \tfrac{\pi}{6}.$$

$$= \left(\underbrace{\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} + 1}_{3/2}\right)\tfrac{\pi}{6}.$$

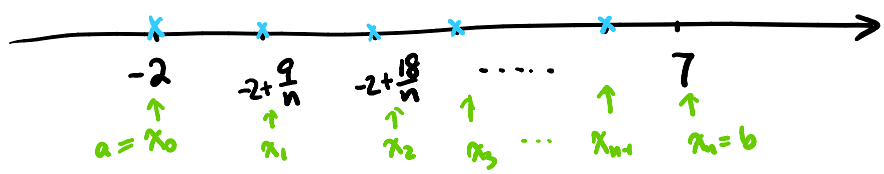
$$\left(\text{or } = \frac{(3+\sqrt{3})\pi}{12}\right)$$

Q7:

$$f(x) = \sqrt[3]{x^3 - x^2 + 9} \quad \text{on } [-2, 7].$$

general approximation : $A \approx L_n$ (for general n.)

$$\Delta x = \frac{b-a}{n} = \frac{7-(-2)}{n} = \frac{9}{n}$$



$$A \approx L_n = \left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right] \cdot \Delta x$$

$$= \left[f(-2) + f\left(-2+\tfrac{9}{n}\right) + f\left(-2+\tfrac{18}{n}\right) + \cdots + f\left(-2+(n-1)\tfrac{9}{n}\right)\right] \cdot \frac{9}{n}$$

① $L_n = \sum_{i=1}^{n} f(\underbrace{a+(i-1)\Delta x}_{\text{left}})\Delta x$

$$= \sum_{i=1}^{n} f\left(-2 + (i-1)\cdot\tfrac{9}{n}\right) \cdot \frac{9}{n}$$

② $R_n = \sum_{i=1}^{n} f(\underbrace{a+i\Delta x}_{\text{right}})\Delta x$

$$= \sum_{i=1}^{n} \sqrt[3]{\left(-2+(i-1)\tfrac{9}{n}\right)^3 - \left(-2+(i-1)\tfrac{9}{n}\right)^2 + 9} \cdot \frac{9}{n}.$$

Q8. $$A = \lim_{n\to\infty} L_n$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+(i-1)\Delta x\right) \cdot \Delta x$$

We know $f(x) = \dfrac{x^2}{3x-2}$, on $[-5, 2]$

$a = -5$ (left endpoint). $\Delta x = \dfrac{b-a}{n} = \dfrac{2-(-5)}{n} = \dfrac{7}{n}$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+(i-1)\Delta x\right) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-5+(i-1)\tfrac{7}{n}\right) \cdot \frac{7}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\left(-5+(i-1)\cdot\frac{7}{n}\right)^2}{3\left(-5+(i-1)\frac{7}{n}\right) - 2} \cdot \frac{7}{n}$$

Q9. $f(x) = \sqrt{\sin(x)}$ on $[0, \pi]$. $(a = 0)$.

$\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{n} = \dfrac{\pi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(i\cdot\tfrac{\pi}{n}\right) \cdot \frac{\pi}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin\left(i\tfrac{\pi}{n}\right)} \cdot \frac{\pi}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1} \sqrt{\sin\left(i\frac{\pi}{n}\right)} \cdot \frac{\pi}{n}$$

Q 10. Given $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\left(\frac{i\pi}{4n}\right)$.

$\Delta x = \frac{\pi}{4n}$ and $\frac{i\pi}{4n} = a + i\Delta x$.

$\Rightarrow$ $a = 0$, We know $\Delta x = \frac{b-a}{n}$

$\Rightarrow$ $b = \frac{\pi}{4}$

function is $f(x) = \tan(x)$ on $[0, \frac{\pi}{4}]$