Week in Review 10

Monday, November 7, 2022 5:01 PM

Q 1. $f(x) = x^4 - 2x^2 + 3$.

(a) Compute $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$

signs of f' :

f is increasing on $(-1, 0)$ and $(1, \infty)$.

f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(b) Based on (a) and 1st derivative test :

f has loc. min at -1, and 1

f has loc. max at 0

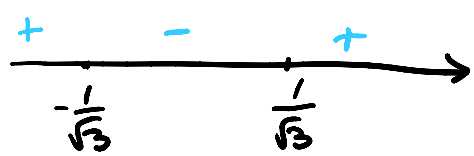
So f loc min value : $f(-1) = (-1)^4 - 2\cdot(-1)^2 + 3 = 2$.

$f(1) = 1^4 - 2\cdot 1^2 + 3 = 2$

f loc max value : $f(0) = 3$.

(c) Compute $f''(x) = 12x^2 - 4 = 4(3x^2 - 1) = 4(\sqrt{3}x - 1)(\sqrt{3}x + 1)$.

signs of f'' :



f is concave up : $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$

concave down : $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

f has inflection points : $(-\frac{1}{\sqrt{3}}, \frac{22}{9})$ and $(\frac{1}{\sqrt{3}}, \frac{22}{9})$.

x-coord value. x-coord. value.

x-coord. value. x-coord. value.

Q 2 $f(x) = x^2 \ln(x)$.

(a) Compute $f'(x) = 2x\ln(x) + x^2 \cdot \frac{1}{x} = 2x\ln(x) + x = x(2\ln(x)+1)$.

$2\ln(x)+1=0$

$\Rightarrow \ln(x) = -\frac{1}{2}$

$\Rightarrow x = e^{-\frac{1}{2}}$

signs of f': DNE $\quad 0 \quad - \quad e^{-1/2} \quad +$

So f is increasing on $(e^{-\frac{1}{2}}, \infty)$

f is decreasing on $(0, e^{-\frac{1}{2}})$

(b) From (a) & 1^{st} derivative test

So f has loc. min at $x = e^{-\frac{1}{2}}$.

loc. min value is $f(e^{-\frac{1}{2}}) = (e^{-\frac{1}{2}})^2 \ln e^{-\frac{1}{2}}$

$= \boxed{-\frac{1}{2} \cdot e^{-1}}$

No loc. max.

(c) Compute: $f''(x) = 2 \cdot \ln(x) + 2x \cdot \frac{1}{x} + 1 = 2\ln(x) + 3$.

$2\ln(x)+3=0$

$\Rightarrow \ln(x) = -\frac{3}{2}$

$x = e^{-3/2}$

Signs of f'': (DNE) $\quad 0 \quad - \quad e^{-3/2} \quad +$

So f is concave up on $(e^{-\frac{3}{2}}, \infty)$

is concave down on $(0, e^{-3/2})$

So inflection point is $(e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3})$

$\uparrow f(e^{-\frac{3}{2}}) = (e^{-\frac{3}{2}})^2 \ln(e^{-3/2})$.

Q3:

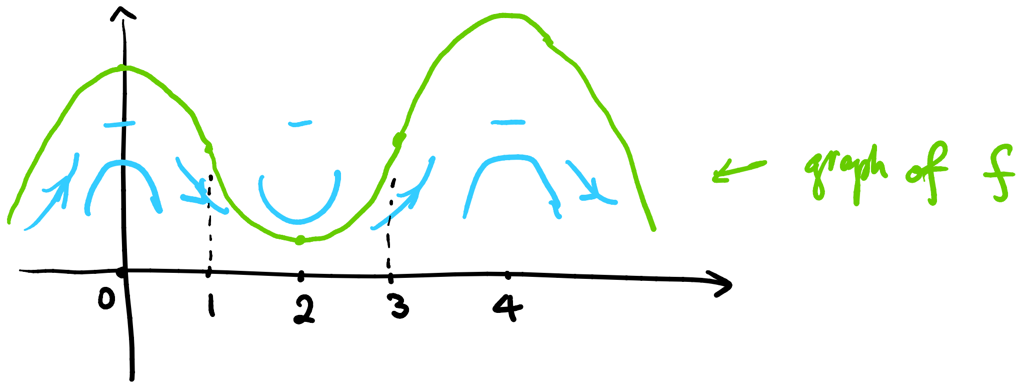
(a) $f'(0) = f'(2) = f'(4) = 0$.

(b) $f' > 0$ $(-\infty, 0)$ and $(2, 4)$.

$f' < 0$ $(0, 2)$ and $(4, \infty)$

(c) $f'' > 0$ $(1, 3)$

$f'' < 0$ $(-\infty, 1)$ and $(3, \infty)$.



Q4.

(a) $f'(5) = 0$.

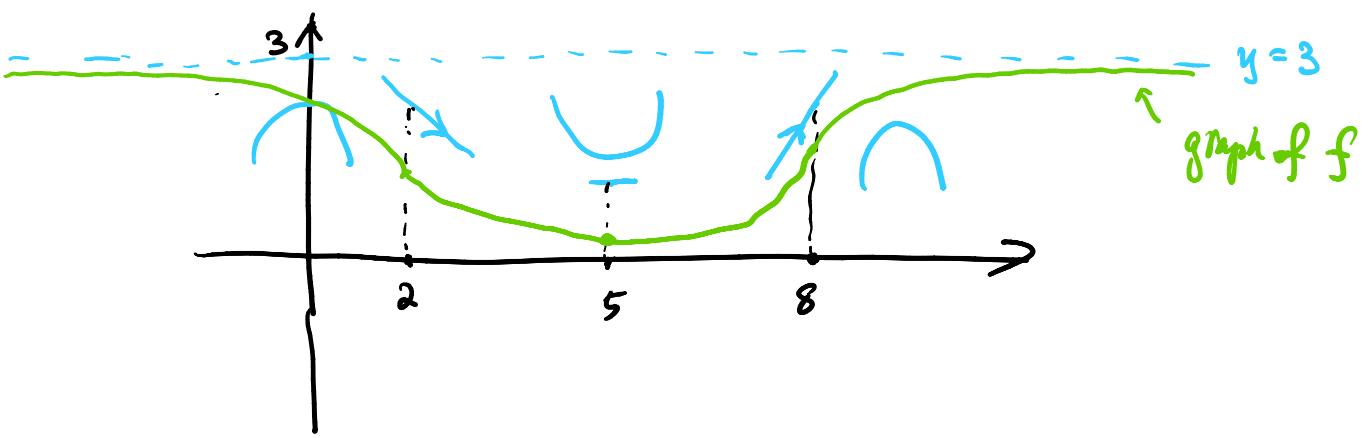
(b) $f' < 0$ $(-\infty, 5)$, $f' > 0$ $(5, \infty)$

(c) $f''(2) = f''(8) = 0$.

(d) $f'' < 0$ $(-\infty, 2)$, $(8, \infty)$

$f'' > 0$ $(2, 8)$

(e) $\lim_{x \to \infty} f(x) = 3$ and $\lim_{x \to -\infty} f(x) = 3$



Q5 $f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$

(a) $\lim_{x \to 0^-} f(x) = -\infty$ vertical asymptote is $x = 0$.

(a) $\lim_{x\to 0^-} f(x) = -\infty$ vertical asymptote is $x=0$.

horizontal asymptote: $\lim_{x\to -\infty} f(x) = 1$ $\lim_{x\to\infty} f(x) = 1$

$y=1$

(b) Compute $f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$

signs of f':

$$\begin{array}{ccccc} - & & + & & - \\ \hline & 0 & & 2 & \end{array}$$

f increasing on $(0,2)$

decreasing on $(-\infty, 0)$ and $(2,\infty)$

(c) Based on (b) $x \neq 0$ (x is not defined at 0). $\Rightarrow$ $x=2$ loc. max (only one).

f has loc. max value $f(2) = 1 + \frac{1}{2} - \frac{1}{2^2} = \frac{5}{4}$.

(d) Compute $f''(x) = \frac{2}{x^3} - \frac{6}{x^4} = \frac{2x-6}{x^4} = \frac{2(x-3)}{x^4}$

sign of f'':

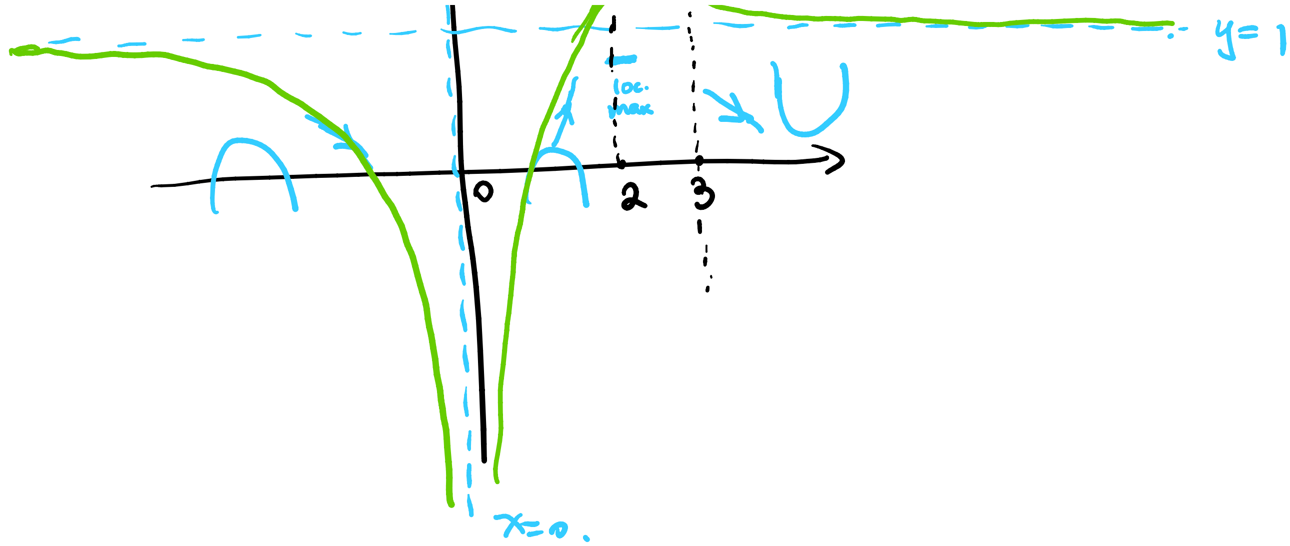
$$\begin{array}{ccccc} - & & - & & + \\ \hline & 0 & & 3 & \end{array}$$

f is concave up $(3,\infty)$

concave down on $(-\infty, 0)$ and $(0,3)$

f has inflection point at $x=3$. $(3, \frac{11}{9})$ (coordinates). $f(3)$.

(e)



Q6.

(a) $$\lim_{x\to 0} \frac{\arccos(x) - \frac{\pi}{2}}{3x} \qquad \text{" } \frac{0}{0} \text{ "}$$

$$= \lim_{x\to 0} \frac{\left(\arccos(x) - \frac{\pi}{2}\right)'}{(3x)'} \qquad \text{(L's rule)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{\sqrt{1-x^2}}}{3} = \boxed{-\frac{1}{3}}$$

(b) $$\lim_{x\to 1} \frac{e^{3x-3} + x^3 - 2}{5\ln(x) + 4x - 4} \qquad \text{" } \frac{0}{0} \text{ "}$$

$$= \lim_{x\to 1} \frac{e^{3x-3} \cdot 3 + 3x^2}{5/x + 4} \qquad \text{(L's rule).}$$

$$= \frac{6}{9} = \boxed{\frac{2}{3}}$$

(c) $$\lim \; x^3 e^{-x^3} \qquad \text{" } \infty \cdot 0 \text{ "}$$

(c) $\lim_{x\to\infty} x^3 e^{-x^3}$ "$\infty\cdot 0$"

$= \lim_{x\to\infty} \frac{x^3}{e^{x^3}}$ "$\frac{\infty}{\infty}$"

$= \lim_{x\to\infty} \frac{3x^2}{e^{x^3}\cdot 3x^2}$ (L's rule).

$= \lim_{x\to\infty} \frac{1}{e^{x^3}} = \boxed{0}$

(d) $\lim_{x\to\infty} x\cdot \sin\left(\frac{\pi}{x}\right)$ "$\infty\cdot 0$"

$= \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$ "$\frac{0}{0}$"

$= \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}}$ (L's rule)

$= \lim_{x\to\infty} \cos\left(\frac{\pi}{x}\right)\cdot \pi = 1\cdot\pi = \boxed{\pi}$

(e) $\lim_{x\to 0^+} x^3 \ln(x)$ "$0\cdot\infty$"

$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x^3}}$ "$\frac{\infty}{\infty}$"

$= \lim_{x\to 0^+} \frac{1/x}{-3/x^4}$ (L's rule)

$= \lim_{x\to 0^+} \frac{x^4/x}{-3} = \lim_{x\to 0^+} \frac{x^3}{-3} = \boxed{0}$

$$= \lim_{x\to 0^+} \frac{x^2/x}{-3} = \lim_{x\to 0^+} \frac{x}{-3} = \boxed{0}$$

(f) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{e^x-1}\right)$ "$\infty - \infty$"

$$= \lim_{x\to 0} \frac{(e^x-1) - x^2}{x^2(e^x-1)}$$ "$\frac{0}{0}$"

$$= \lim_{x\to 0} \frac{e^x - 2x}{2x(e^x-1) + x^2e^x}$$ (L's rule).

$$= \frac{e^0}{\underbrace{2\cdot 0(e^0-1)}_{\substack{+\cdot+ \\ -\cdot-}} + \underbrace{0^2 e^0}_{+}} = \frac{1}{\underbrace{0\cdot 0}_{+} + \underbrace{0^2\cdot 1}_{+}} = \infty$$ (>0).

(g) $\lim_{x\to\infty} \left(\frac{2x^2}{2x+1} - \frac{x^2}{x+3}\right)$. "$\infty - \infty$"

$$= \lim_{x\to\infty} \frac{2x^2(x+3) - x^2(2x+1)}{(2x+1)(x+3)}$$

$$= \lim_{x\to\infty} \frac{\cancel{2x^3} + 6x^2 - \cancel{2x^3} - x^2}{(2x+1)(x+3)}$$

$$= \lim_{x\to\infty} \frac{5x^2}{(2x+1)(x+3)} = \boxed{\frac{5}{2}}$$

(h) $\lim_{x\to 0^+} (3x+1)^{\csc(x)}$ "1^∞"

$$= \lim_{x\to 0^+} e^{\ln (3x+1)^{\csc(x)}}$$

$$= \lim_{x\to 0^+} e^{\ln(3x+1)^{\csc(x)}}$$

$$= \lim_{x\to 0^+} e^{\csc(x)\ln(3x+1)}$$

$$= e^{\lim_{x\to 0^+} \frac{1}{\sin(x)} \cdot \ln(3x+1)} \qquad \text{"} \frac{0}{0} \text{"}$$

$$= e^{\lim_{x\to 0^+} \frac{\frac{1}{3x+1}\cdot 3}{\cos(x)}}$$

$$= e^{3/1} = \boxed{e^3}$$

(i) $\lim_{x\to\infty} (1+x+x^2)^{\frac{1}{\ln(x)}}$ "∞^0"

$$= \lim_{x\to\infty} e^{\ln(1+x+x^2)^{\frac{1}{\ln(x)}}}$$

$$= \lim_{x\to\infty} e^{\frac{1}{\ln(x)}\ln(1+x+x^2)} \qquad \text{"} \frac{\infty}{\infty} \text{"}$$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+x+x^2}\cdot(1+2x)}{\frac{1}{x}}} \qquad \text{(L's rule)}$$

$$= e^{\lim_{x\to\infty} \frac{x(1+2x)}{1+x+x^2}}$$

$$= e^{\frac{2}{1}} = \boxed{e^2}$$

(j) $\lim \left(1+\frac{3}{x}\right)^{5x}$ "1^∞"

(j) $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{5x}$ "1^∞"

$= \lim_{x\to\infty} e^{\ln\left(1+\frac{3}{x}\right)^{5x}}$

$= \lim_{x\to\infty} e^{5x\ln\left(1+\frac{3}{x}\right)}$ "$\infty \cdot 0$"

$= e^{\lim_{x\to\infty} 5\cdot \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}}$ "$\frac{0}{0}$"

$= e^{\lim_{x\to\infty} 5\cdot \frac{\frac{1}{1+\frac{3}{x}}\cdot\left(-\frac{3}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}}}$

$= e^{\lim_{x\to\infty} 5\cdot \frac{3}{1+\frac{3}{x}}} = \boxed{e^{15}}$