Spring 2012 Math 151

$y' = ky$

$y = Ce^{kx}$ / $y = Ca^x$

Week in Review # 8

sections: 4.5, 4.6, 4.8

courtesy: Joe Kahlig

1. A curve passes through the point $(\overset{x}{2}, \overset{y}{10})$ and has the property that the slope of the curve at every point P is three times the y-coordinate of P. Find the equation of this curve.

$y' = 3y \longleftrightarrow y = Ce^{3x}$

$10 = Ce^{6}$

$\frac{10}{e^6} = C \rightarrow C = 10e^{-6}$

$$\boxed{\begin{aligned} y &= 10e^{-6}e^{3x} \\ y &= 10e^{3x-6} \end{aligned}}$$

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2. A bacteria culture starts with 800 bacteria and will have 1000 bacteria after 30 minutes. Assume that the culture grows at a rate proportional to the number of bacteria present.

$y' = ky$

(a) Find a formula for the number of bacteria after t ~~hours~~ min.

(b) Find the number of bacterial after 1 day.

(c) When will the population reach 3200 bacteria.

$$y = C_0 e^{kt}$$
$$\boxed{y = 800 e^{kt}}$$
$$1000 = 800 e^{30k}$$
$$\frac{10}{8} = e^{30k}$$
$$1.25 = e^{30k}$$
$$\ln(1.25) = \ln e^{30k} = 30k$$
$$k = \frac{\ln(1.25)}{30} = .007438$$

$$y = C_0 a^t$$
$$\boxed{y = 800 a^t}$$
$$1000 = 800 a^{30}$$
$$1.25 = a^{30}$$
$$a = (1.25)^{1/30}$$
$$a = 1.00744585$$

b) $y(1440) = 35,873,240.69$

$24(60)$

c)
$$y = 800 e^{kt}$$
$$3200 = 800 e^{kt}$$
$$4 = e^{kt}$$
$$\ln 4 = \ln e^{kt}$$
$$\ln(4) = kt$$
$$t = \frac{\ln(4)}{k} = 186.377 \text{ min}$$

found in part A)

$$4 = a^t$$
$$\ln(4) = \ln(a^t)$$
$$\ln(4) = t\ln(a)$$
$$\frac{\ln(4)}{\ln(a)} = t$$

3. A chemical has a half-life of 18 days. A sample is obtained and 5 days later there remains 50 grams of the chemical.

   (a) Find a formula that will give the amount of the chemical that remains t days after the sample is obtained.

   (b) What was the initial amount of the sample of this chemical?

   (c) How long will it take until 70% of sample is gone?

$$y = C_0 e^{kt}$$

$$\frac{1}{2} C_0 = C_0 e^{18k}$$

$$.5 = e^{18k}$$

$$\ln(.5) = \ln e^{18k}$$

$$\ln(.5) = 18k$$

$$\frac{1}{18}\ln(.5) = k$$

$$k = -.038508$$

$$50 = C_0 e^{5k}$$

b) $C_0 = \dfrac{50}{e^{5k}} = 60.616\text{g}$

a) $y = 60.616 e^{kt}$

$$y = 100 e^{kt}$$

$$30 = 100 e^{kt}$$

$$.3 = e^{kt}$$

$$\ln(.3) = \ln e^{kt}$$

$$\ln(.3) = kt$$

$$t = \frac{\ln(.3)}{k} = 31.265 \text{ days.}$$

$$.30(60.616) = 60.616 e^{kt}$$

4. A turkey is taken from a $350°F$ oven into a room with a temperature of $80°F$. Fifteen minutes later, the turkey is $250°$. Use Newton's Law of cooling to solve this problem.

(a) Find a formula that will give the temperature of the turkey at time t.

(b) What will the temperature be after 40 minutes?

$$y' = K(A - y) \rightarrow y = A + Ce^{Kt}$$

Temp Room (A) Temp of object (y)

$A = 80$ $(0, 350)$ $(15, 250)$

$350 = 80 + Ce^{0}$

$270 = C$

$$\boxed{y = 80 + 270e^{Kt}}$$

$250 = 80 + 270e^{15K}$

$170 = 270e^{15K}$

$\frac{17}{27} = e^{15K}$

$\ln\left(\frac{17}{27}\right) = \ln\left(e^{15K}\right) = 15K$

$K = \frac{1}{15}\ln\left(\frac{17}{27}\right)$

$= -.0308415$

b) $y(40) = 158.63°F$

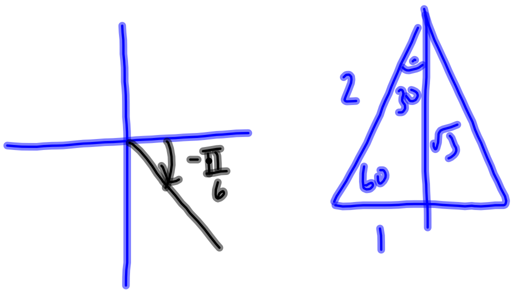
$y = \arcsin(x) = \sin^{-1}(x)$ means $\sin y = x$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ $\quad -1 \le x \le 1$

$y = \arccos(x) = \cos^{-1}(x)$ means $\cos y = x$ and $0 \le y \le \pi$ $\quad -1 \le x \le 1$

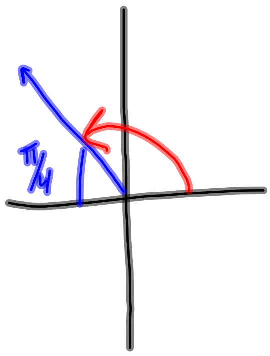
$y = \arctan(x) = \tan^{-1}(x)$ means $\tan y = x$ and $\frac{-\pi}{2} < y < \frac{\pi}{2}$ $\quad$ any Real #

5. Find the exact value of the following without the aid of a calculator.

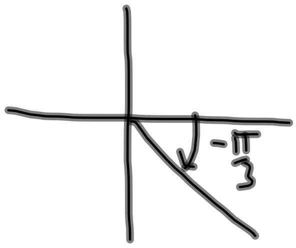
(a) $\sin^{-1}\left(\frac{-1}{2}\right) = -\frac{\pi}{6}$



(b) $\arccos\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$



(c) $\arctan(-\sqrt{3}) =$ $-\frac{\pi}{3}$



(d) $\tan(\arctan(1.25)) =$ 1.25

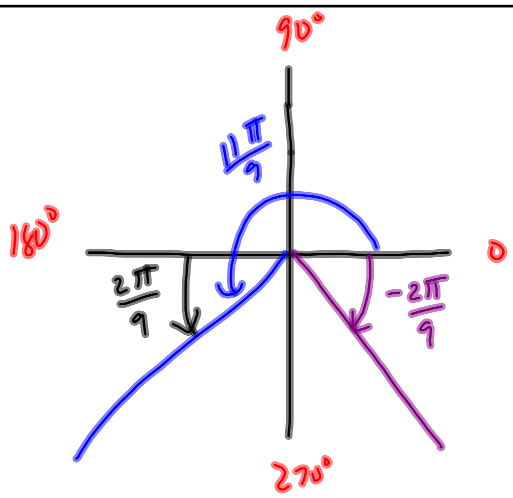
(e) $\sin\left(\arcsin\left(\frac{\pi}{2}\right)\right) =$ DNE

$\frac{\pi}{2} > 1$

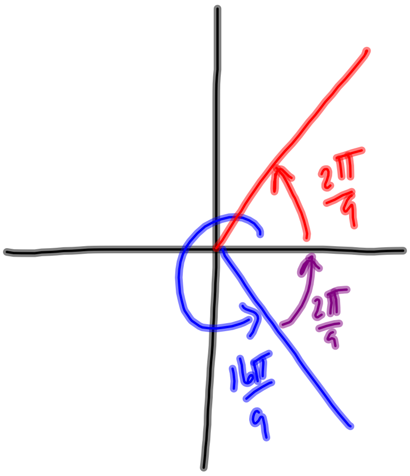
$\sin(\arcsin(.7)) = .7$

(f) $\sin^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right) =$ $\frac{\pi}{2}$

(g) $\sin^{-1}\left(\sin\left(\frac{11\pi}{9}\right)\right) = \quad -\frac{2\pi}{9}$



(h) $\arccos\left(\cos\left(\frac{16\pi}{9}\right)\right) = \quad \frac{2\pi}{9}$



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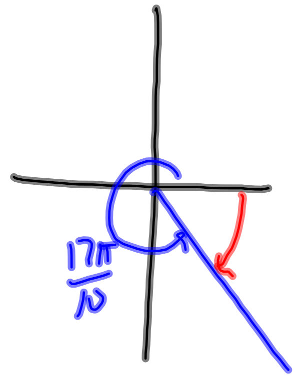
(i) $\arctan\left(\tan\left(\frac{17\pi}{10}\right)\right) = -\frac{3\pi}{10}$

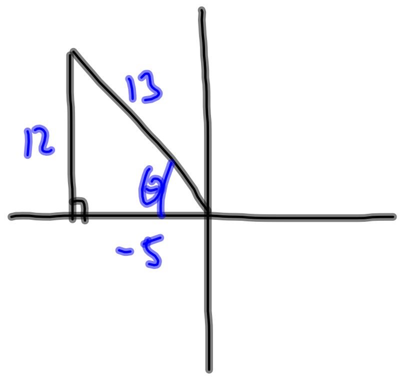
$\frac{\pi}{10} \to 18^\circ$

$$\begin{array}{r} {}^{5} \\ 18 \\ \times\ 7 \\ \hline 126 \end{array}$$

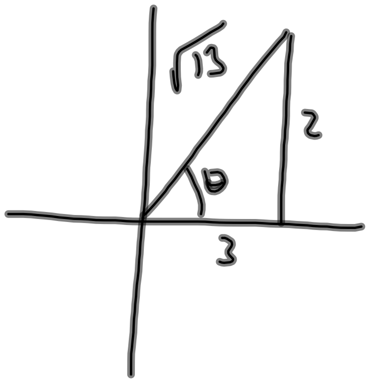


(j) $\tan\left(\underbrace{\arccos\left(\frac{-5}{13}\right)}\right) = \frac{12}{-5} = -\frac{12}{5}$

$\cos\theta = \frac{-5}{13}$



(k) $\sec\left(\arctan\left(\frac{2}{3}\right)\right) = \frac{\sqrt{13}}{3}$



(l) $\sin\left(2\arctan\left(\frac{2}{3}\right)\right) = 2\sin\Big(\underbrace{\arctan\left(\tfrac{2}{3}\right)}_{\theta}\Big)\cos\Big(\underbrace{\arctan\left(\tfrac{2}{3}\right)}_{\theta}\Big)$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

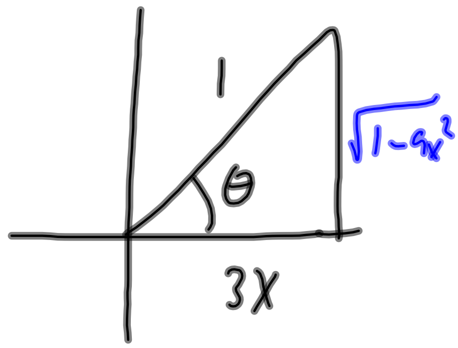
$$= 2\left(\frac{2}{\sqrt{13}}\right)\cdot\left(\frac{3}{\sqrt{13}}\right) = \frac{12}{13}$$

6. Write $\tan(\cos^{-1} 3x)$ without any trig functions.

$$\cos^{-1}(3x) = \theta$$

$$\cos(\theta) = \frac{3x}{1}$$

$$\tan(\theta) = \frac{\sqrt{1-9x^2}}{3x}$$



Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

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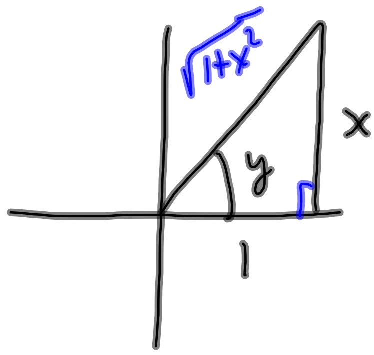
7. Prove the derivative rule for $y = \tan^{-1}(x)$.



$$y = \tan^{-1}(x)$$

$$\boxed{\tan(y) = x}$$

$$\sec^2(y) \cdot y' = 1$$

$$y' = \frac{1}{\sec^2(y)} = \frac{1}{\left(\frac{\sqrt{1+x^2}}{1}\right)^2} = \frac{1}{1+x^2}$$

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8. Find the derivatives of the following.

(a) $y = \tan^{-1}(5x)$

$$y' = \frac{1}{1+(5x)^2} \cdot 5 = \frac{5}{1+25x^2}$$

(b) $y = x^2 \arcsin(x^2)$

$$y' = 2x \arcsin(x^2) + x^2 \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$= 2x \arcsin(x^2) + \frac{2x^3}{\sqrt{1-x^4}}$$

(c) $y = (\cos^{-1}(7x))^3$

$$y' = 3\left(\cos^{-1}(7x)\right)^2 \cdot \frac{-1}{\sqrt{1-(7x)^2}} \cdot 7$$

$$= \frac{-21\left(\cos^{-1}(7x)\right)^2}{\sqrt{1-49x^2}}$$

L'Hopital's Rule $\frac{0}{0}, \frac{\infty}{\infty}$

9. $\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{-\sin(x)}{6x} \overset{\frac{0}{0}}{\underset{L'H}{=}}$

$= \lim_{x\to 0} \frac{-\cos(x)}{6} = \boxed{\frac{-1}{6}}$

10. $\lim_{x\to\infty} \dfrac{\ln(x+e^{3x})}{2x}$ $\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$ $\lim_{x\to\infty} \dfrac{\dfrac{1+3e^{3x}}{x+e^{3x}}}{\underset{1}{2}} = \lim_{x\to\infty} \dfrac{1+3e^{3x}}{2x+2e^{3x}}$ $\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$

$= \lim_{x\to\infty} \dfrac{9e^{3x}}{2+6e^{3x}}$ $\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$ $\lim_{x\to\infty} \dfrac{27e^{3x}}{18e^{3x}} = \lim_{x\to\infty} \dfrac{27}{18} = \dfrac{27}{18}$

$= \boxed{\dfrac{3}{2}}$

$\frac{2}{0}$

11. $\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2} = +\infty$

If $\lim_{x\to 0} \frac{e^x + e^{-x}}{x^3} = \text{DNE}$

$\lim_{x\to 0^+} \frac{e^x + e^{-x}}{x^3} \neq \lim_{x\to 0^-} \frac{e^x + e^{-x}}{x^3}$

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$\infty - \infty$

12. $\displaystyle \lim_{x\to\infty} \left(\frac{x^2}{x-1} - \frac{x^2}{x+5} \right)$

$\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

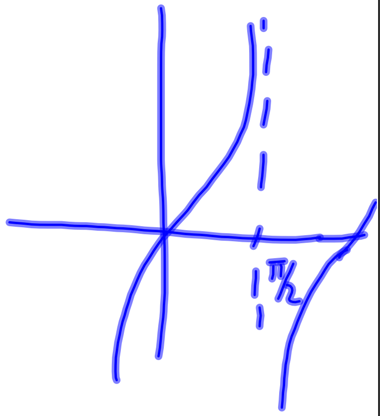
$$\lim_{x\to\infty} \frac{x^2(x+5) - x^2(x-1)}{(x-1)(x+5)}$$

$$\lim_{x\to\infty} \frac{x^3+5x^2-x^3+x^2}{x^2+4x-5} = \lim_{x\to\infty} \frac{6x^2}{x^2+4x-5}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{12x}{2x+4} \overset{L'H}{=} \lim_{x\to\infty} \frac{12}{2} = \frac{12}{2} = 6$$

$0 \cdot \infty$ $\qquad \frac{\infty}{\infty}$ or $\frac{0}{0}$

13. $\lim_{x\to\frac{\pi}{2}} (2x-\pi)\tan(x)$



$\frac{0}{0}$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{2x-\pi}{\cot(x)} \overset{L'H}{=}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{2}{-\csc^2(x)} = \lim_{x\to\frac{\pi}{2}^-} -2\cdot\sin^2(x) = -2(1)^2 = -2$$

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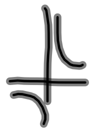
0^0

$1^\infty,\ 0^0,\ \infty^0$

14. $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{\ln x^x} = \lim\limits_{x\to 0^+} e^{\boxed{x\ln(x)}} = e^0 = 1$

need to know $\quad 0\cdot\infty \quad \frac{\infty}{\infty} \quad \frac{1}{x}$



$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-1x^{-2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -\frac{1}{x}\cdot\frac{x^2}{1}$$

$$= \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x = 0$$

1^{∞}

15. $\lim_{x \to 0^+} (1-5x)^{\frac{1}{x}} = \lim_{x \to 0^+} e^{\ln(1-5x)^{1/x}} = \lim_{x \to 0^+} e^{\boxed{\frac{1}{x}\ln(1-5x)}}$

$= e^{-5}$

$\frac{0}{0}$

$$\lim_{x \to 0^+} \frac{\ln(1-5x)}{x} = \lim_{x \to 0^+} \frac{\frac{-5}{1-5x}}{1} = -5$$