Spring 2012 Math 151

Sample Problems for Exam 3
sections: 4.2–4.6, 4.8, 5.1–5.3, 5.5, and 5.7
courtesy: Joe Kahlig

$\sqrt[3]{x^2} = x^{2/3}$

1. Find $f(x)$.

(a) $f'(x) = x\sqrt[3]{x^2} + \dfrac{7}{x^2} + 3$ $= x^{5/3} + 7x^{-2} + 3$

$f(x) = \frac{3}{8}x^{8/3} - 7x^{-1} + 3x + C$

(b) $f'(x) = \dfrac{x-8}{4x^2}$ $= \dfrac{x}{4x^2} - \dfrac{8}{4x^2} = \dfrac{1}{4}\cdot\dfrac{1}{x} - \dfrac{2}{x^2}$

$f' = \frac{1}{4}x^{-1} - 2x^{-2}$

$f = \frac{1}{4}\ln|x| + 2x^{-1} + C$

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(c) $f'(x) = (x^2 - 3)^2 = (x^2-3)(x^2-3) = x^4 - 6x^2 + 9$

$f(x) = \frac{x^5}{5} - \frac{6x^3}{3} + 9x + C$

(d) $f'(x) = \sec(x)\tan(x) + 2sec^2(x)$

$f(x) = \sec(x) + 2\tan(x) + C$

2. Find the function $f(x)$ if $f''(x) = 8\sin(x) + e^{2x}$ if $f(0) = 6$ and $f'(0) = 5$

$$f'(x) = -8\cos(x) + \frac{1}{2}e^{2x} + C$$

$$f'(0) = 5 = -8 + \frac{1}{2} + C$$

$$5 = -7.5 + C$$

$$12.5 = C$$

$$f'(x) = -8\cos(x) + \frac{1}{2}e^{2x} + 12.5$$

$$f(x) = -8\sin(x) + \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + 12.5x + K$$

$$f(0) = 6 = 0 + \frac{1}{4} + 0 + K$$

$$6 = \frac{1}{4} + K$$

$$5.75 = K$$

$$\boxed{f(x) = -8\sin(x) + \frac{1}{4}e^{2x} + 12.5 + 5.75}$$

$$y = e^{kx}$$
$$y' = ke^{kx}$$

$$f' = e^{kx}$$
$$f = \frac{1}{k}e^{kx}$$

3. Find $f'(x)$.

(a) $y = \log_3(x^2+1) + 5^{x^3}$

$$y' = \frac{2x}{(x^2+1)\ln(3)} + 3x^2 \cdot 5^{x^3} \cdot \ln(5)$$

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(b) $y = \arctan(5x) + \ln\sqrt{x^4+1}$ $= \arctan(5x) + \frac{1}{2}\ln(x^4+1)$

$$y' = \frac{1}{1+(5x)^2}\cdot 5 + \frac{1}{2}\,\frac{4x^3}{x^4+1}$$

$$= \frac{5}{1+25x^2} + \frac{2x^3}{x^4+1}$$

(c) $y = \sin^{-1}(3x^2)$

$$y' = \frac{1}{\sqrt{1-(3x^2)^2}}\cdot 6x$$

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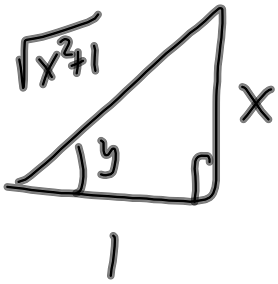
$y = \arctan(x)$

$$\boxed{\tan y = x}$$



$$\sec^2(y) \cdot y' = 1$$

$$\sec y = \sqrt{x^2+1}$$

$$y' = \frac{1}{\sec^2(y)} = \frac{1}{x^2+1}$$

(d) $y = \left(\dfrac{x^2+2}{x^4+1}\right)^{5x}$

$$\ln y = \ln\left(\frac{x^2+2}{x^4+1}\right)^{5x}$$

$$\ln(y) = 5x \ln\left(\frac{x^2+2}{x^4+1}\right)$$

$$\ln(y) = 5x \cdot \left[\ln(x^2+2) - \ln(x^4+1)\right]$$

$$\frac{y'}{y} = 5\ln\left(\frac{x^2+2}{x^4+1}\right) + 5x \cdot \left[\frac{2x}{x^2+2} - \frac{4x^3}{x^4+1}\right]$$

$$\boxed{y' = \left(\frac{x^2+2}{x^4+1}\right)^{5x} \cdot \left[5\ln\left(\frac{x^2+2}{x^4+1}\right) + 5x\left(\frac{2x}{x^2+2} - \frac{4x^3}{x^4+1}\right)\right]}$$

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4. Find the value of A so that $c = 2$ will satisfy the conclusion of the Mean Value Theorem on the interval $[0, A]$ for $f(x) = x^3 + x - 1$.

on $[0, A]$ there is a # c with $0 < c < A$

such that

$$f'(c) = \frac{f(A) - f(0)}{A - 0}$$

$f'(x) = 3x^2 + 1$

$f'(c) = f'(2) = 13$

$$13 = \frac{f(A) - f(0)}{A - 0} = \frac{A^3 + A - 1 - (-1)}{A}$$

$$13 = \frac{A^3 + A}{A}$$

$$13 = \frac{A(A^2 + 1)}{A}$$

$$13 = A^2 + 1$$

$$12 = A^2$$

$$A = \pm\sqrt{12} \rightarrow A = \sqrt{12}$$

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5. Find the inverse function for $y = \dfrac{1+2^x}{3-2^x}$

$$x = \frac{1+2^y}{3-2^y}$$

$$x(3-2^y) = 1+2^y$$

$$3x - x2^y = 1+2^y$$

$$3x-1 = x2^y + 2^y$$

$$3x-1 = (x+1)2^y$$

$$\frac{3x-1}{x+1} = 2^y$$

either answer

$$\boxed{y = \log_2\left(\frac{3x-1}{x+1}\right)}$$

$$\boxed{\frac{1}{\ln(2)} \ln\left(\frac{3x-1}{x+1}\right) = y}$$

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6. If $f(x) = 4 + x + e^{3x-6}$ is one to one and $g(x)$ is the inverse of $f(x)$, then compute $g'(7)$.

$f(g(x)) = x$

$f'(g(x)) \cdot g'(x) = 1$

$g'(x) = \dfrac{1}{f'(g(x))}$

$g(7) = \underline{\ 2\ }$

$f(\underline{\ 2\ }) = 7$

$f' = 1 + 3e^{3x-6}$

$g'(7) = \dfrac{1}{f'(g(7))} = \dfrac{1}{f'(2)} = \dfrac{1}{1+3} = \dfrac{1}{4}$

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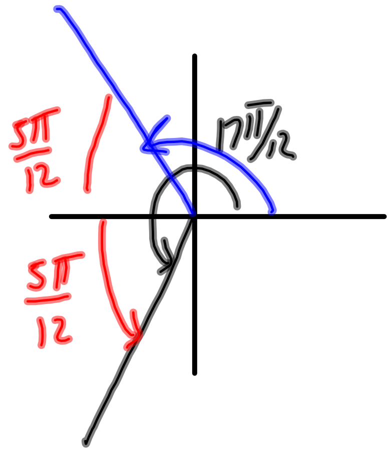
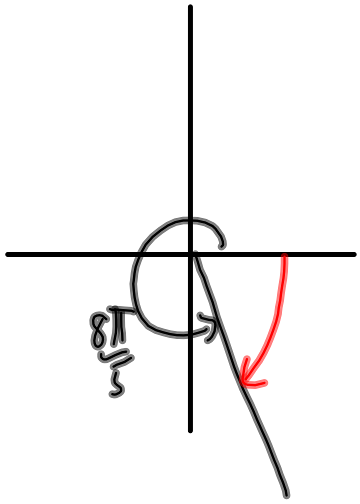
7. Compute the exact value.

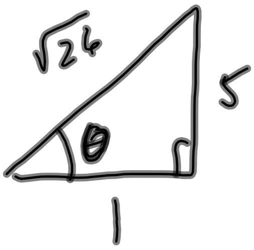
(a) $\arccos\left(\cos\left(\frac{17\pi}{12}\right)\right) =$ $\frac{7\pi}{12}$



(b) $\arcsin\left(\sin\left(\frac{8\pi}{5}\right)\right) =$ $-\frac{2\pi}{5}$



(c) $\cos(\tan^{-1}(5)) =$ $\frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$



$\theta = \tan^{-1}(5)$

$\tan\theta = 5$

$+\infty - +\infty$ $\qquad x\ln(x) - x + 1$

8. Compute the exact values of these limits.

(a) $\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right) = \lim_{x\to 1^+}\left(\frac{x\ln(x) - 1(x-1)}{(x-1)\ln(x)}\right)$

$$= \lim_{x\to 1^+} \frac{1\ln(x) + x\cdot\frac{1}{x} - 1}{\ln(x) + (x-1)\cdot\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\ln(x)}{\ln(x) + 1 - \frac{1}{x}} = \lim_{x\to 1^+} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

(b) $\lim\limits_{x\to 0} (\cos(3x))^{-1/x^2} = \lim\limits_{x\to 0} e^{\ln(\cos(3x))^{\frac{-1}{x^2}}} = e^{9/2}$

$$\lim_{x\to 0} \frac{-1}{x^2}\ln(\cos(3x)) = \lim_{x\to 0} \frac{-\ln(\cos(3x))}{x^2}$$

$$\lim_{x\to 0} \frac{-\frac{-\sin(3x)\cdot 3}{\cos(3x)}}{\frac{2x}{1}} = \lim_{x\to 0} \frac{3\sin(3x)}{2x\cos(3x)}$$

$$\lim_{x\to 0} \frac{3\cos(3x)\cdot 3}{2\cos(3x) + 2x\cdot(-\sin(3x)\cdot 3}$$

$$\lim_{x\to 0} \frac{9\cos(3x)}{2\cos(3x) - 6x\sin(3x)} = \boxed{\frac{9}{2}}$$

9. A population has the characteristic that at any given time(in years) its growth rate is three times the size of the population. At the end of the first year the size of the population is 70. Find a formula that gives the population where x is the number of years since the population started.

$y' = 3y \longrightarrow y = Ce^{3x}$

$y(1) = 70$

$70 = Ce^{3}$

$\frac{70}{e^3} = C \qquad C = 70e^{-3}$

$y = 70e^{-3}e^{3x}$

$\boxed{y = 70e^{3x-3}}$

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10. Solve for x. $\log x + \log(x+6) = \log(2x-4)$

$$\log x(x+6) = \log(2x-4)$$

$$x^2+6x = 2x-4$$

$$x^2+4x+4 = 0$$

$$x = \frac{-4 \pm \sqrt{16-4(1)(4)}}{2} = \frac{-4}{2} = -2$$

has no solution.

not in the domain

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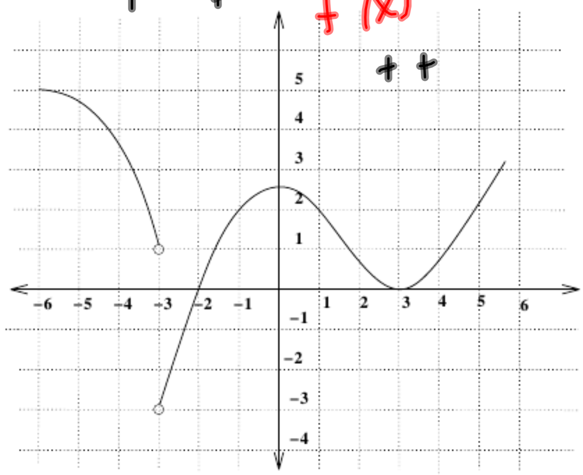
11. Here is the **graph of the derivative, f′(x),** of a function $f(x)$. Assume that the function $f(x)$ is continuous for all real numbers. Use the graph to answer the following questions.

(a) Give the critical values of $f(x)$ and then tell which are local maximums, local minimums, or neither.

+ - + $f'(x)$



$f'=0$

Rel min @

$x=-2$

$x=3$ ← not a max or min

f' DNE

$x=-3$ Rel max. at

(b) Find the intervals where $f(x)$ is decreasing and the intervals where it is increasing.

Inc $(-\infty,-3) \cup (-2,3) \cup (3,\infty)$

Dec $(-3,-2)$

$f(x)$ c.u. $\Leftrightarrow f''(x) > 0 \Leftrightarrow f'$ inc.

$f(x)$ c.d. $\Leftrightarrow f'' < 0 \Leftrightarrow f'$ dec.

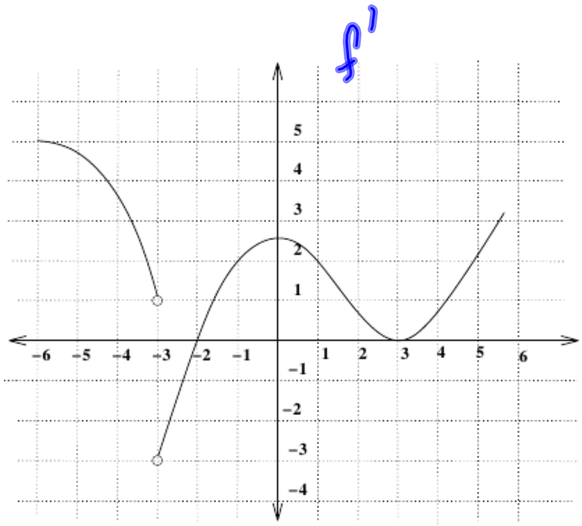
(c) Find the intervals where $f(x)$ is concave down and the intervals where it is concave up.

concave up

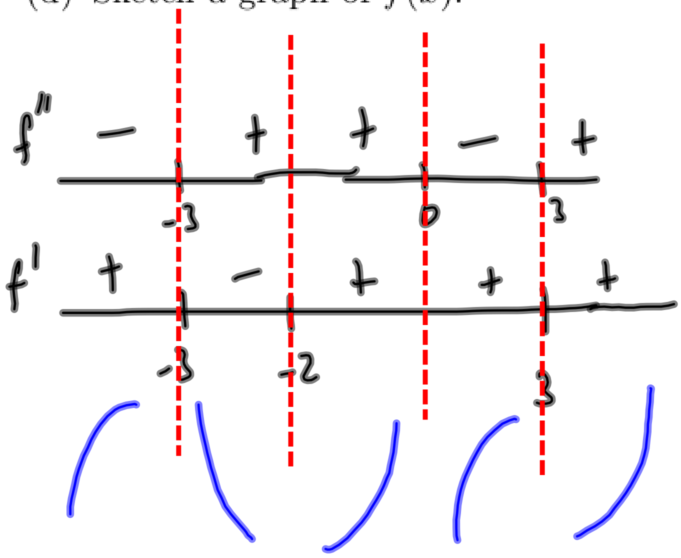
$(-3,0) \cup (3,\infty)$

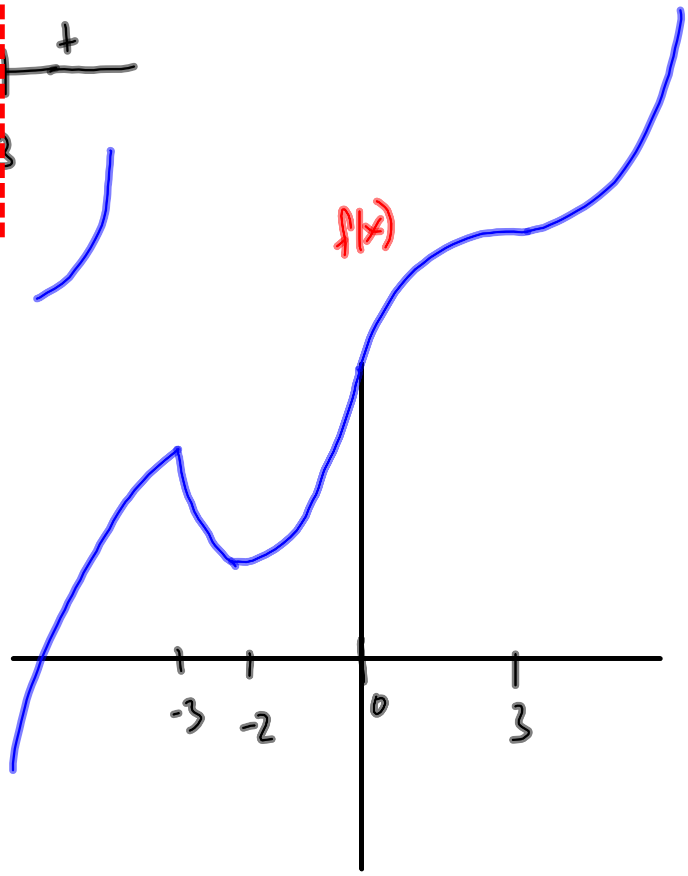
concave down

$(-\infty,-3) \cup (0,3)$



(d) Sketch a graph of $f(x)$.





12. Use the function $f(x) = \dfrac{1}{x(x-3)^2}$ to answer the following questions.

(a) Find the intervals where the function is increasing and the intervals where it is decreasing.

(b) Find the relative extreme of $f(x)$.

(c) Sketch a graph of $f(x)$.

Domain is all Real #s except $x=0$ + $x=3$

$f' = \dfrac{-3x+3}{x^2(x-3)^3}$

Set $f'=0$ + solve.

$0 = \dfrac{-3x+3}{x^2(x-3)^3}$

$0 = -3x+3$

$3x = 3$

$x = 1$ ← C.V.

f': $-$ | $-$ | $+$ | $-$ at 0 (V.A), 1 (C.V.), 3 (V.A)

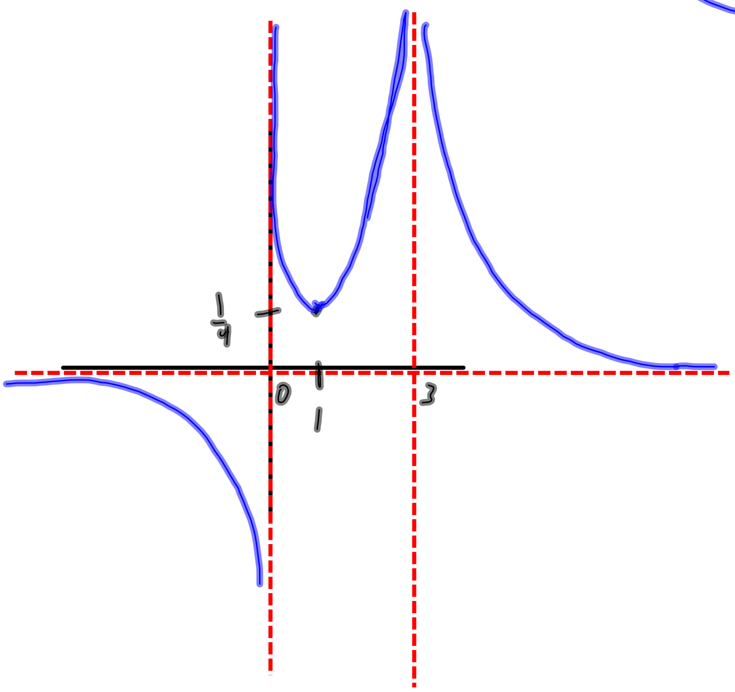
HA $y=0$

$\lim_{x\to\infty} f(x)$

$\lim_{x\to-\infty} f(x)$

A) Inc $(1,3)$ Dec $(-\infty,0) \cup (0,1) \cup (3,\infty)$

B) Rel. min @ $x=1$ → Rel min $= \frac{1}{4}$



C.U. $(0,3) \cup (3,\infty)$

C.D. $(-\infty, 0)$

$$f = \frac{1}{x(x-3)^2}$$

$$f' = \frac{0 - 1\cdot\left[1(x-3)^2 + x\cdot 2(x-3)\right]}{\left(x(x-3)^2\right)^2}$$

$$= \frac{-(x-3)^2 - 2x(x-3)}{x^2(x-3)^4} = \frac{(x-3)\left[-(x-3) - 2x\right]}{x^2(x-3)^4}$$

$$= \frac{-3x+3}{x^2(x-3)^3}$$

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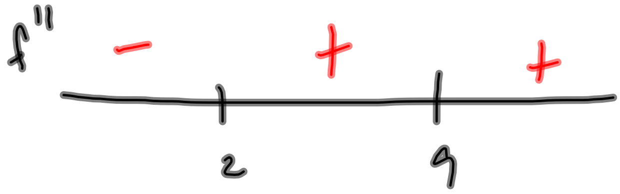
13. The domain of $f(x)$ is all real numbers. Find the intervals where $f(x)$ is concave up and where it is concave down. Give the x-values of the inflection points.

$$f''(x) = (x-9)^2(x-2)e^{(2x^3-6)}$$

$f'' = 0 \longrightarrow x = 2, 9$



c.u.
$(2,9) \cup (9,\infty)$

c.d
$(-\infty, 2)$

Inflection pt @ $x = 2$

14. Find the absolute extreme values for the function on the given interval.

(a) $f(x) = x^3 - 12x + 12$ on $[-3, 1]$

cont function ✓

closed interval ✓

$f' = 3x^2 - 12$

$0 = 3x^2 - 12$

$12 = 3x^2$

$4 = x^2$

$x = \pm 2$

$f(-3) = -27 + 36 + 12 = 21$

$f(-2) = -8 + 24 + 12 = 28$

$f(1) = 1 - 12 + 12 = 1$

$\boxed{\begin{array}{l}\text{abs max} = 28 \\ \text{abs min} = 1\end{array}}$

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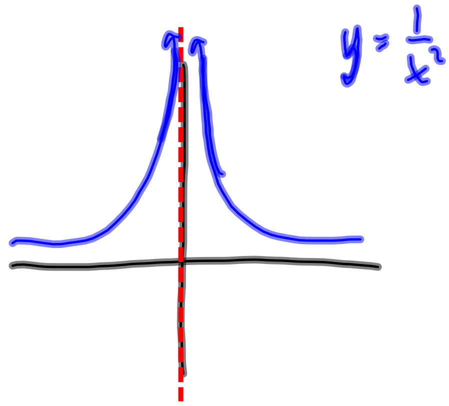
$= \frac{1}{x^2}$

(b) $g(x) = x^{-2}$ on $\underline{[-2, 1]}$

cont? → NO

$g' = -2x^{-3}$

V.A. @ $x = 0$



$g' = 0$

$\frac{-2}{x^3} = 0$

no solution

so no C.V.

no abs max.

$g(-2) = \frac{1}{4}$ ← abs min

$g(1) = 1$

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15. Find the values of a and b so that $f(x) = ax^2 + b\ln(x)$ will have an inflection point at $(e^2, 5)$.

$f''(e^2) = 0$

$f' = 2ax + \frac{b}{x}$

$f'' = 2a - \frac{b}{x^2}$

$0 = 2A - \frac{B}{(e^2)^2}$

$0 = 2A - \frac{B}{e^4}$

$\frac{B}{e^4} = 2A$

$B = 2Ae^4$

$f(e^2) = 5$

$5 = A(e^2)^2 + B\ln(e^2)$

$5 = Ae^4 + 2B$

$5 = Ae^4 + 2(2Ae^4)$

$5 = Ae^4 + 4Ae^4$

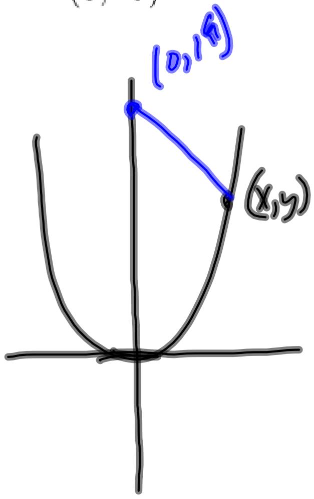
$5 = 5Ae^4$

$1 = Ae^4$

$\boxed{e^{-4} = A}$

$\boxed{B = 2(e^{-4})e^4 = 2}$

16. Find the point(s) on the graph of $y = \frac{x^2}{2}$ that is the closest to the point $(0, 19)$.



$$d = \sqrt{(x-0)^2 + (y-19)^2}$$

$$f = d^2 = (x-0)^2 + (y-19)^2$$

$$f = x^2 + \left(\frac{x^2}{2} - 19\right)^2$$

$$f = x^2 + \frac{x^4}{4} - 19x^2 + 361$$

$$f = \frac{x^4}{4} - 18x^2 + 361$$

$$f' = x^3 - 36x \longrightarrow f'' = 3x^2 - 36$$

$$0 = x(x^2 - 36)$$

$$0 = x \qquad x = \pm 6$$

$f''(0) < 0$ Rel max

$f''(6) > 0$ Rel min

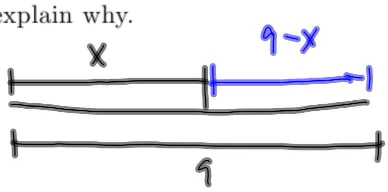
$f''(-6) > 0$ Rel min

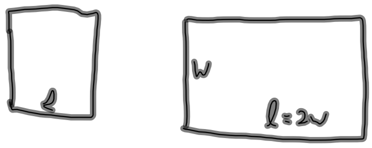
Answers

$(6, 18)$

$(-6, 18)$

17. A wire 9 m long is to be cut into at most 2 pieces. The first piece, x, is bent into a square; the second piece is bent into a rectangle whose length is twice the width. How much of the wire should be used for the square so that the total area enclosed is a maximum? If this is not possible, then explain why.





$$s = \frac{x}{4}$$

$$w + w + 2w + 2w = 9 - x$$
$$6w = 9 - x$$
$$w = \frac{9-x}{6}$$
$$l = \frac{9-x}{3}$$

$$A = s^2 + lw$$
$$= \left(\frac{x}{4}\right)^2 + \left(\frac{9-x}{3}\right)\left(\frac{9-x}{6}\right) = \frac{x^2}{16} + \frac{81 - 18x + x^2}{18}$$

$$A = \frac{x^2}{16} + \frac{81}{18} - x + \frac{x^2}{18}$$

domain
$0 \le x \le 9$

$$A' = \frac{2x}{16} - 1 + \frac{2x}{18} = \frac{x}{8} - 1 + \frac{x}{9}$$

$$A'' = \frac{1}{8} + \frac{1}{9} > 0$$

any c.v. will be a min

so all the wire is for the square or red.

$$A(0) = \frac{81}{18}$$
$$A(9) = \left(\frac{9}{4}\right)^2 = \frac{81}{16} \leftarrow \text{bigger so use}$$

all of the wire for the square.