

## Quadratic Functions

**Definition:** If  $a$ ,  $b$ ,  $c$ ,  $h$ , and  $k$  are real numbers with  $a \neq 0$ , then the functions

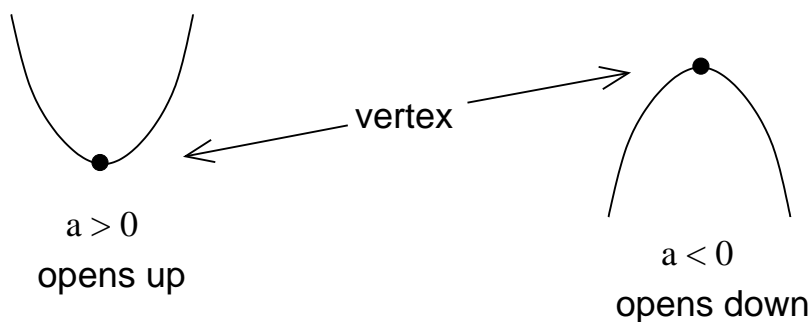
$$y = ax^2 + bx + c \quad \text{standard form}$$

$$y = a(x - h)^2 + k \quad \text{vertex form}$$

both represent a **quadratic function**. The graph of a quadratic function is called a **parabola**.

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In both of the above formulas, the value of  $a$  determines if the graph opens upward ( $a > 0$ ) or opens downward ( $a < 0$ ).



The **vertex** of a parabola is the point on the graph that has either the highest (when opening down) or the lowest (when opening up)  $y$ -value. This is the point where the graph, in essence, changes direction. The  $y$ -value of the vertex is called the minimum value of the quadratic when the graph opens up or the maximum value of the quadratic when the graph opens down.

The vertex is easy to find when the formula is given in vertex form. It is the point  $(h, k)$ . If the formula is in standard form, then the  $x$ -coordinate of the vertex is found by  $x = \frac{-b}{2a}$ . To find the  $y$ -coordinate of the point, plug in this  $x$ -value into the formula.

**Note:** There is a method called *completing the square* that will transform a quadratic function written in standard form into vertex form. This method will not be taught in this course since it is easier to find the vertex with the  $x = \frac{-b}{2a}$  computation.

The  $x$ -intercepts of the quadratic function can be found by setting the quadratic equation equal to zero and solving for  $x$ . This can be done in a variety of ways. If the quadratic function is in the standard form, then you can use the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \text{provided } b^2 - 4ac \geq 0$$


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**Example 1:** Use the quadratic function  $y = 2x^2 - 16x + 45$  to answer the following.

- Determine if the function opens upward or downward.
- Find the vertex of the parabola.
- Find the maximum value.

- (d) Find the minimum value.
- (e) Find the  $x$ -intercepts.

**Solution:**

- (a) Opens upward since  $a = 2 > 0$ .
- (b) The  $x$ -value of the vertex is found by  $x = \frac{-b}{2a} = \frac{-(-16)}{2 * 2} = 4$ .

The  $y$ -value of the vertex is  $y = 2(4)^2 - 16(4) + 45 = 13$ .  
Vertex: (4, 13)

- (c) No maximum since the parabola opens upward.
- (d) Minimum value is 13.
- (e) No  $x$ -intercepts since the vertex is above the  $x$ -axis and the parabola opens up. This can also be shown using the quadratic formula.

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 * 2 * 45}}{2 * 2} = \frac{16 \pm \sqrt{-104}}{4}$$

There is no solution since the formula has the square root of a negative number.

**Example 2:** Use the quadratic function  $y = -5(x + 3)^2 - 15$  to answer the following.

- (a) Determine if the function opens upward or downward.
- (b) Find the vertex of the parabola.
- (c) Find the maximum value.
- (d) Find the minimum value.
- (e) Find the  $x$ -intercepts.

**Solution:**

- (a) Opens downward since  $a = -5 < 0$ .
- (b) The formula was given in vertex form:  
 $y = -5(x + 3)^2 - 15 = -5(x - (-3))^2 + (-15)$ .  
Thus  $h = -3$  and  $k = -15$ .  
Vertex: (-3, -15)
- (c) Maximum value is -15.
- (d) No minimum value since the parabola opens downward.
- (e) No  $x$ -intercepts since the vertex is below the  $x$ -axis and the parabola opens downward.

**Example: 3:** Use the quadratic function  $y = 2x^2 + 4x - 198$  to answer the following.

- (a) Determine if the function opens upward or downward.
- (b) Find the vertex of the parabola.
- (c) Find the maximum value.
- (d) Find the minimum value.
- (e) Find the  $x$ -intercepts.

**Solution:**

(a) Opens upward since  $a = 2 > 0$ .

(b) The  $x$ -value of the vertex is found by  $x = \frac{-b}{2a} = \frac{-(4)}{2 * 2} = -1$ .

The  $y$ -value of the vertex is  $y = 2(-1)^2 + 4(-1) - 198 = -200$ .

Vertex:  $(-1, -200)$

(c) No maximum since the parabola opens upward.

(d) Minimum value is  $-200$ .

(e) Using the quadratic formula

$$x = \frac{-4 \pm \sqrt{4^2 - 4 * 2 * (-198)}}{2 * 2} = \frac{-4 \pm \sqrt{1600}}{4}$$

$$x = 9 \text{ or } x = -11$$

$x$ -intercepts:  $(9, 0)$  and  $(-11, 0)$ .

In Section 1.3 and 1.4 we worked with revenue functions where every item,  $x$ , was sold at a fixed price,  $p$ . Thus, the formula of the revenue function is  $R(x) = p * x$ .

In these sections we also discussed the demand function, also called the price-demand function. In the price-demand function, the price of the item varies based on the number of items demanded(bought) by consumers. Combining these two concepts gives us revenue functions that are not linear. If the price-demand function is linear, then the revenue function will be a quadratic function.

**Example 4:** Find the formula for the revenue function if the price-demand function of a product is  $p = 54 - 3x$ , where  $x$  is the number of items sold and the price is in dollars. How many items should be sold in order to maximize the revenue? What is the maximum revenue?

**Solution:**

$$R(x) = p * x = (54 - 3x) * x = 54x - 3x^2 = -3x^2 + 54x$$

Since  $a = -3$  indicates the revenue function opens downward, there will be a maximum and it will be at the vertex. The number of items needed to maximize the revenue function is

$$x = \frac{-b}{2a} = \frac{-54}{2 * (-3)} = 9 \text{ items}$$

The maximum revenue is  $R(9) = -3(9)^2 + 54 * 9 = \$243$  dollars.

**Example 5:** A company has determined that if the price of an item is \$40, then 150 will be demanded by consumers. When the price is \$45, then 100 items are demanded by consumers.

- Find the price-demand equation, assuming that it is linear.
- Find the revenue function.
- Find the number of items sold that will give the maximum revenue. What is the maximum revenue?
- What is the price of each item when maximum revenue is achieved?
- If the company has a fixed cost of \$3520 and a variable cost of \$12 per item, find the company's linear cost function.
- Find the company's profit function.
- Find the number of items sold that will give the maximum profit. What is the maximum profit?
- How many items should be sold for the company to break even?

**Solution:**

- (a) Compute the equation of the line using the points (150, 40) and (100, 45)

$$m = \frac{45 - 40}{100 - 150} = -0.1$$

$$p - 40 = -0.1(x - 150)$$

$$\text{price-demand function: } p = -0.1x + 55$$

- (b)  $R(x) = p * x = (-0.1x + 55) * x = -0.1x^2 + 55x$

- (c) Since  $a = -0.1$  the revenue function opens downward and will have a maximum. The revenue is maximized when  $x = \frac{-b}{2a} = \frac{-55}{2 * (-0.1)} = 275$  items are sold.

$$\text{The maximum revenue is } R(275) = -0.1(275)^2 + 55 * 275 = \$7562.50.$$

- (d) The price of each item when the revenue is maximized is found by plugging the  $x$ -value of the vertex into the price-demand function.

$$p = -0.1 * 275 + 55 = \$27.50.$$

- (e) Cost function:  $C(x) = 12x + 3520$

- (f) Profit function = Revenue - Cost

$$P(x) = -0.1x^2 + 55x - (12x + 3520) = -0.1x^2 + 43x - 3520$$

- (g) Since  $a = -0.1$  the profit function opens downward and will have a maximum. The profit is maximized when  $x = \frac{-b}{2a} = \frac{-43}{2 * (-0.1)} = 215$  items are sold.

$$\text{The maximum profit is } P(215) = -0.1(215)^2 + 43 * 215 - 3520 = \$1102.50.$$

- (h) The company will break even when profit is zero:

$$P(x) = -0.1x^2 + 43x - 3520 = 0$$

$$x = \frac{-43 \pm \sqrt{43^2 - 4 * (-0.1) * (-3520)}}{2(-0.1)} = \frac{-43 \pm \sqrt{441}}{-0.2}$$

$$x = 110 \text{ or } x = 320 \text{ items}$$

## Exercises

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In Problems 1-6 find the vertex, the maximum value, the minimum value, and the  $x$ -intercepts of the quadratic.

1.  $y = 4x^2 - 56x + 96$

4.  $y = -4x^2 + 40x - 122$

2.  $y = -3x^2 + 24x + 99$

5.  $y = 5(x - 3)^2 - 125$

3.  $y = 2x^2 - 40x + 225$

6.  $y = -3(x + 1)^2 - 12$ 

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7. A study found that the yield (in thousands of pounds of vegetables),  $y$ , was related to the amount of fertilizer per acre (in hundreds of pounds),  $x$ , by  $y = 1 + 1.08x - 0.3x^2$ .

(a) How much fertilizer per acre would you use to maximize the yield? Round your the answer to the nearest pound.

(b) What is the maximum yield?

8. A deli sells 640 sandwiches per day at a price of \$8 each. A market survey shows that for every \$0.10 reduction in price, 40 more sandwiches will be sold.

(a) Find the linear price-demand function.

(b) Find the revenue equation.

(c) How many sandwiches should be sold to maximize the revenue?

(d) How much should the deli charge for a sandwich in order to maximize its revenue?

(e) What is the maximum revenue for the deli?

9. An apartment complex has a demand given by  $p = -5x + 860$ , where  $x$  is the number of apartments rented and  $p$  is the price for each apartment, in dollars. The cost of maintaining the complex is \$120 per rented apartment. There are fixed costs of \$6000.

(a) Find the revenue equation.

(b) Find the cost equation.

(c) Find the profit equation.

(d) Find the number of apartments to be rented that maximize profit.

(e) Find the price that the apartments are rented at when the profit is maximized.

10. A stadium has found that if ticket prices are \$10 then 3000 people come to the game. If the tickets are priced at \$5, then 8000 people come to the game. Assume that this relationship is linear.

(a) Find the demand equation for tickets to the game.

(b) Find the revenue equation.

(c) Find how many people need to buy tickets in order to maximize revenue.

(d) Find the maximum revenue.

11. A manufacturer of a digital camera provided the information that when the price per camera is \$89, then 1 million cameras will sell. When the price of each camera is \$79, then 3 million cameras will sell.
- Find a linear demand function where  $x$  is the number of millions of cameras sold and  $p$  is the price of each camera in dollars.
  - What is the company's revenue function for this camera?
  - Find the number of cameras that would be needed to be sold to maximize the revenue for the company.
  - At what price is the company selling the cameras at when the revenue is maximized?
12. The manufacturer of digital cameras in Problem 11 has provided the cost of producing and selling  $x$  million cameras is  $C(x) = 19x + 156.25$ .
- What is the company's profit function?
  - How many cameras must be produced and sold so that the company will have a maximum profit?
  - What is the price of the cameras when the company is maximizing their profit?
  - How many cameras would the company have to produce and sell to break even?

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## Answers

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- vertex  $(7, -100)$   
maximum value: none  
minimum value:  $-100$   
 $x$ -intercepts:  $(2, 0)$  and  $(12, 0)$
- vertex  $(4, 147)$   
maximum value:  $147$   
minimum value: none  
 $x$ -intercepts:  $(-3, 0)$  and  $(11, 0)$
- vertex  $(10, 25)$   
maximum value: none  
minimum value:  $25$   
 $x$ -intercepts: none
- vertex  $(5, -22)$   
maximum value:  $-22$   
minimum value: none  
 $x$ -intercepts: none
- vertex  $(3, -125)$   
maximum value: none  
minimum value:  $-125$   
 $x$ -intercepts:  $(-2, 0)$  and  $(8, 0)$
- vertex  $(-1, -12)$   
maximum value:  $-12$   
minimum value: none  
 $x$ -intercepts: none

7. (a) 180 pounds of fertilizer  
 (b) 1,972 pounds of vegetables
8. (a) Compute the equation of the line using the points (640, 8) and (680, 7.90).

$p = -0.0025x + 9.6$  where  $x$  is the number of sandwiches sold and  $p$  is the price of each sandwich in dollars.

- (b)  $R(x) = p * x = (-0.0025x + 9.6) * x$   
 $R(x) = -0.0025x^2 + 9.6x$
- (c)  $x = \frac{-b}{2a} = \frac{-9.6}{2 * (-.0025)} = 1920$  sandwiches
- (d) price =  $-0.0025 * 1920 + 9.6 = \$4.80$
- (e) \$9,216

9. (a)  $R(x) = p * x = (-5x + 860) * x$   
 $R(x) = -5x^2 + 860x$
- (b)  $C(x) = 120x + 6000$
- (c)  $P(x) = R(x) - C(x)$   
 $P(x) = -5x^2 + 860x - (120x + 6000)$   
 $P(x) = -5x^2 + 740x - 6000$
- (d)  $x = \frac{-b}{2a} = \frac{-740}{2 * (-5)} = 74$

74 apartments

- (e) price =  $-5 * 74 + 860 = \$490$

10. (a) Compute the equation of the line using the points (3000, 10) and (8000, 5).

$p = -0.001x + 13$  where  $x$  is the number of tickets sold and  $p$  is the price of each ticket in dollars.

- (b)  $R(x) = p * x = (-0.001x + 13) * x$   
 $R(x) = -0.001x^2 + 13x$
- (c) 6500 people
- (d) \$42,250

11. (a) Compute the equation of the line using the points (1, 89) and (3, 79).

$p = -5x + 94$

- (b)  $R(x) = p * x = (-5x + 94) * x$   
 $R(x) = -5x^2 + 94x$
- (c) 9.4 million cameras
- (d) \$47

12. (a)  $P(x) = R(x) - C(x)$   
 $P(x) = -5x^2 + 94x - (19x + 156.25)$   
 $P(x) = -5x^2 + 75x - 156.25$
- (b) 7.5 million cameras
- (c) \$56.50
- (d) break even at 2.5 million cameras or 12.5 million cameras